THE STABILITY OF PARTICLE-LIKE SOLUTIONS OF SOME NON-LINEAR FIELD EQUATIONS

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The Stability of Particle-like

Solutions of Some Nonlinear

Field Equations

A Thesis presented by

David Lessells Thomson Anderson

to the

University of St. Andrews

in application for the Degree of

Doctor of Philosophy

September 1969.

Declaration

I hereby declare that the accompanying Thesis is my own composition, that it is based upon research carried out by me, and that no part of it has previously been presented in application for a Higher Degree.

Certificate

I certify that David L. T. Anderson has spent nine terms as a research student in the Department of Theoretical Physics of the United College of St. Salvator and St. Leonard in the University of St. Andrews, that he has fulfilled the conditions of Ordinance No. 16 of the University Court of St. Andrews and that he is qualified to submit the accompanying Thesis in application for the Degree of Doctor of Philosophy.

Research Supervisor.

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Finally my gratitude to my wife for her unfailing moral support particularly during the latter stages of preparation of this Thesis.

Career

In 1966, I graduated from the University of St. Andrews with a first class honours degree in Physics with Theoretical Physics. In October of that year I was admitted by the Senatus Academicus of the University of St. Andrews as a Research Student. I received financial support from the Carnegie Trust for the Universities of Scotland during the whole period of my study for this Thesis.

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It is the customary fate of new truths to begin as heresies and to end as superstitions.

Thomas Huxley.

APPENDICES

Abstract

The object of this thesis is to examine the stability of particle-like solutions of the nonlinear field equation

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \kappa^2\psi - \mu^2\psi\psi^*\psi + \lambda(\psi\psi^*)^2\psi$$

with the particular form of time-dependence

$$\psi = \varphi(r)\, e^{-i\omega t}$$

Initially our interest is concentrated on the case $\lambda = 0$. We begin the analysis by finding spherically symmetric particle-like solutions, and then examining the stability of the lowest-order solution by first-order perturbation theory. Direct perturbation methods are then considered. This solution is found to be highly unstable whether it is time-independent ($\omega = 0$) or not ($\omega \neq 0$).

The more general case $\lambda \neq 0$ is next discussed. Particle-like solutions are found to exist in this case for

$$-\infty < \lambda(\kappa^2 - \omega^2/c^2)/\mu^4 < 3/16$$

On examining the stability of the lowest-order solutions of this generalised field equation, it is found that for correct choice of the field parameters stable time-dependent solutions can exist, some of which can also have the attractive feature that their energy density is positive definite.

We conclude by considering some methods of extending the theory.

1. INTRODUCTION

1.1 Introduction

One of the main problems confronting physicists today, is that of understanding elementary particles. What are the basic building blocks of matter, what are elementary particles, why do they exist? The name elementary must surely be a misnomer since they are not elementary, and might even not be fundamental. It appears that what was originally hoped would be a small family, is now not small, and indeed as one goes to higher and higher energy, more and more 'particles' are discovered. It is true that the later discoveries have been of particles which live for very short lengths of time and whose true existence might well be questioned, but there is no apparent end to the process.

Because of the basic importance of the subject, considerable work has been done on it, both theoretical and experimental. Theoretically quantum field theory has had success in describing the interactions between particles, and group theory in classifying the particles in symmetry groups. Few would claim that either approach is truely satisfactory but nevertheless considerable progress has been made even to the extent of predicting the existence of hitherto unobserved particles. For example, the Ω^- was predicted before it was found experimentally.

Conventional field theory is unsatisfactory in many respects. Too many fields exist. Whenever a new particle is discovered a new field is required. Not only this, but interaction terms must be added/

added to the Hamiltonian incorporating all interactions with other existing particles. e.g. if one has only the existence of the electromagnetic field and the proton field, then one has a Hamiltonian.

$$H = H_{em} + H_p + H_{int}\ (em : p)$$

The addition of a new particle, say a π meson requires the Hamiltonian to be modified to

$$H = H_{em} + H_p + H_{\pi} + H_{int}(em : p) + H_{int}(em : \pi) + H_{int}(p : \pi)$$

In view of the large number of 'particles' known to date, the full Hamiltonian, is an extremely complicated expression. It would be nicer if one could have a single universal field Ψ containing all the known particles, and a single universal field equation governing Ψ's behaviour.

In some ways a more serious defect of conventional field theory is the occurrence of divergences. For example, on evaluating the energy of an electron in an external radiation field, one finds that it is infinite. This difficulty is overcome by mass renormalisation. One can consider the electron not as a point singularity but as having some spatial extension. The problem then arises as to what distribution one should take for the electron. This approach requires one to have two electron masses, the bare electron mass and the observed electron mass and is generally considered to be unsatisfactory.

Any sensible theory must be nonlinear. A linear theory can not give interaction between particles which is in contradiction to the laws of nature, for particles do interact. So a linear theory must be modified and interaction terms inserted. When one inserts such interaction/

interaction terms the field equation becomes nonlinear. Why not try to reverse the process, choose the field equation to be nonlinear and extract the interaction? That it is possible to do this was shown by Einstein, Infeld, and Hoffman[1] who were able to derive the equations of motion of gravitating bodies from the nonlinear field equations, to show that Einstein's Field equations imply that particles move along geodesics.

To overcome some of the above difficulties, we are led to postulate a new type of field theory to represent elementary particles with the requirements that it shall be free from divergences or singularities, and that the field equations shall be, à priori, nonlinear. The requirement of Lorentz invariance seems a reasonable one, though even this has latterly been called to question. In this work, only classical fields are considered.

Although we have said that we hope to use the field to represent elementary particles, we have not yet indicated how this should be done in a nonlinear field theory. Perhaps the best physical representation is in terms of the energy density. We picture a space of elementary particles as a space in which the energy density has distinct local maxima. These local concentrations of energy correspond to elementary particles. In general the dynamical evolution of the system will be determined by the governing field equation. By following the time development of this equation, one should be able to find out how the particles behave: how they interact, whether they are stable or not, whether they radiate energy, whether they coalesce to form new particles, or/

or decay into other particles.

As stated earlier we hope that it may eventually prove possible to represent particles by a universal field. This is, however, far too ambitious a programme for the moment, and instead we will look at as simple a nonlinear classical Lorentz invarient field with particle-like solutions as we can find. (In accordance with what has previously been said, we define a particle-like solution to be one which is singularity free, mathematically well behaved, and has associated with it finite energy, distributed in such a way as to give rise to some local concentration(s) of energy, which can be taken to represent particle(s)).

In conventional field theory it is generally accepted that the Klein-Gordon equation represents a spinless particle, and that if one wants to represent a spin $\frac{1}{2}$ particle one should consider the Dirac equation. In this theory it is by no means certain that a nonlinear modification of the Dirac equation is the correct way to represent a spin $\frac{1}{2}$ particle. Spin may be some manifestation of a non spherically symmetric particle-like solution, obeying a one component field equation such as a nonlinear modification of the Klein-Gordon equation. In this present work we shall consider only spherically symmetric particle-like solutions to some modifications of the Klein-Gordon equation and thus, in any case, restrict ourselves to spinless particles.

A simple way to modify the Klein-Gordon equation is to add to it some polynomial in ψ, denoted $f(\psi, \psi^*)$, yielding an equation of/

of the form

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = K^2\psi + f(\psi, \psi^*)$$

The particular form $f(\psi, \psi^*) = -\mu^2 \psi\psi^*\psi + \lambda\, \psi\psi^*\psi\psi^*\psi$ will be examined. Initially in chapters 2 - 5 we consider the special case $\lambda = 0$, and then in chapters 6 - 9 treat the more general problem $\lambda \neq 0$. Before so doing, a review of other work in the subject will be given in the next section.

1.2 Historical Survey

For more than fifty years there has been interest in fields which satisfy nonlinear differential equations. The most celebrated example of this is undoubtedly Einstein's General Theory of Relativity[2], but also included are the attempts by Mie and others[3] to modify Maxwell's Equations, and attempts to find models for elementary particles[(4 . . 11)]. Finkelstein and alii[4] examined a nonlinear spinor field, and in the course of their investigation considered a particular modification of the Klein-Gordon equation, showing the existence of particle-like solutions to it.

The theory was put on a firm footing when Enz showed that stable particle-like solutions could exist at least in a world with one space dimension and that these solutions possessed certain symmetry and topological properties. Enz considered the 1-dimensional equation

$$\frac{\partial^2\theta}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\theta}{\partial t^2} = \frac{K}{2}\sin 2\theta$$

showing that it has solutions

$$\theta = \pm \sin^{-1}(\operatorname{sech}\sqrt{K}x)$$

for which the energy density differs essentially from zero in a region of the X axis of length $x_0 = \pi/\sqrt{K}$, and for which the total integrated energy can be expressed as $E = 4\sqrt{K}$. Exact two-particle solutions to this one-dimension field have also been found by Seeger and Kochendorfer[5] and Perring and Skyrme. As a result it has been possible to visualize the collision of two particles in this case, as well as deducing the force law between particles at large distances.

In the 3 dimensional case, one can get particle-like solutions to certain nonlinear modifications of the Klein-Gordon equation but Hobart and Derrick[6] have shown independently that if these solutions are time-independent then they can not be stable. (In the one-dimensional case one can have stable time-independent particle-like solutions). Rosen[7] has found an exact one particle solution to a particular nonlinear field. The field equation

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -3g\psi^5$$

has a solution $\psi = Z/(Z^4g+r^2)^{\frac{1}{2}}$ where Z is an arbitrary constant. Although the field must be unstable, a measure of the decay time can be made which might correspond to the half-life of some unstable elementary particle. For the above field, Rosen finds that the one-particle solution decays in a time $\sim Z^2g^{\frac{1}{2}}/c$

The problem of deducing the force law between particles is more difficult in the three-dimensional case. The field equations should still/

still be sufficient in a well behaved theory to determine the interaction between particles, but whereas in the one-dimensioanl field described it was possible to find an exact two-particle solution and deduce the force law from that, there are no known two-particle solutions to any field in three space dimensions, and so in this case one must resort to approximation methods. Rosen and Rosenstock[8] deduced the interaction between particles in a nonlinear modification of the Klein-Gordon equation. They showed that an equation of the type

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = K^2\psi - \mu^2(\psi\psi^*)^n\psi$$

implied that particles interacted via a Yukawa-type interaction at large distances. Their method of solution was to integrate the normal component of the stress-energy tensor over a large sphere enclosing only one particle and so obtain the force on that particle due to the other.

The force law for Rosen particles has been examined by Rosen, Derrick, Pinski[9]. In these latter studies, while one does not have an exact two-particle state, one does have an exact one-particle state. For two particles, far apart and moving only slowly, the field near each particle should be closely approximated by the form $Z\,(Z^4 g + r^2)^{-1/2}$ The method then consists of finding a variational approximation to the two-particle state, using the above result as a guide to what variational function to choose. It is found that like particles (i.e. same sign of Z) attract each other according to an inverse square law.

1.3 Objectives

In three space dimensions, the Hobart-Derrick theorems rule out the existence of time-independent stable particle-like solutions to nonlinear/

nonlinear modifications of the Klein-Gordon equation. However, little is known about the stability of time-dependent particle-like solutions to such equations. In this work we will examine the stability of some time-dependent particle-like solutions to some nonlinear field equations.

Initially the field equation

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = K^2\psi - \mu^2\psi\psi^*\psi \qquad (1.1)$$

is considered. Later a more general field equation having (1.1) as a special case will be considered.

The equation (1.1) is chosen because it is one of the simplest nonlinear modifications of the Klein-Gordon equation and because it has been shown to have particle-like solutions [4,8,10]

Equation (1.1) can be derived from the variational principle $\delta\int \mathcal{L}\, d^3r\, dt$ with the Lagrangian density $\mathcal{L}$ given by

$$\mathcal{L} = \frac{1}{c^2}\left|\frac{\partial\psi}{\partial t}\right|^2 - \left|\nabla\psi\right|^2 - K^2\left|\psi\right|^2 + \frac{1}{2}\mu^2\left|\psi\right|^4 \qquad (1.2)$$

In the linear (Klein-Gordon) case, the constant K is equal to $mc/\hbar$ where, at least on quantisation, m can be identified with the mass of the particle being represented. In this theory m can not be so identified, and in consequence K is considered to be merely a parameter with no particular physical significance. We are primarily interested in time-dependent solutions, and the form of (1.1) suggests a time-dependence of the form

$$\psi = \varphi(\underset{\sim}{r})\, e^{-i\omega t} \qquad (1.3)$$

where ω is a real parameter. For simplicity $\varphi(r)$ is chosen to/

to be a real spherically symmetric function of r. Then (1.1) can be reduced to an equation involving only one variable r

viz $$\frac{d^2\varphi}{dr^2} + \frac{2}{r}\frac{\partial\varphi}{\partial r} = \left(K^2 - \frac{\omega^2}{c^2}\right)\varphi - \mu^2\varphi^3 \qquad (1.4)$$

It is a necessary condition that φ and its cartesian derivatives exist everywhere, so we require $\left(\frac{d\varphi}{dr}\right)_{r=0} = 0$. Further φ must be particle-like and so φ must tend to zero as r $\to \infty$.

The coordinate transformations

$$r' = r\left(K^2 - \omega^2/c^2\right)^{\frac{1}{2}}$$
$$\varphi' = \mu\varphi\left(K^2 - \omega^2/c^2\right)^{-\frac{1}{2}} \qquad (1.5)$$

bring (1.4) into the parameterless form

$$\frac{d^2\varphi'}{dr'^2} + \frac{2}{r'}\frac{d\varphi'}{dr'} = \varphi' - \varphi'^3 \qquad (1.6)$$

Solutions to (1.6) will be discussed in the next chapter.

In field theory it is customary to take for the energy density

$$\mathcal{E} = -\mathcal{L} + \left(\frac{\partial\psi}{\partial t}\right)\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial t}\right)} + \left(\frac{\partial\psi^*}{\partial t}\right)\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi^*}{\partial t}\right)} \qquad (1.7)$$

This has the property that the total energy is conserved. i.e. the integral of $\mathcal{E}$ over all space remains a constant. If one accepts this definition of energy density then for the above field, this takes the particular form

$$\mathcal{E} = \frac{1}{c^2}\left|\frac{\partial\psi}{\partial t}\right|^2 + \left|\underset{\sim}{\nabla}\psi\right|^2 + K^2|\psi|^2 - \frac{1}{2}\mu^2|\psi|^4 \qquad (1.8)$$

There is no gain in generality in inserting a multiplicative constant into (1.2), (1.8), since such a constant can be absorbed into ψ with a corresponding redefinition of μ.

Charge/

Charge and current densities have standard definitions in field theory. The quantities

$$\rho = -i\sigma\left(\psi\frac{\partial\psi^*}{\partial t} - \psi^*\frac{\partial\psi}{\partial t}\right) \qquad \underset{\sim}{J} = -i\sigma c^2\left(\psi^*\underset{\sim}{\nabla}\psi - \psi\underset{\sim}{\nabla}\psi^*\right) \tag{1.9}$$

obey the conservation law

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{J} + \frac{\partial\rho}{\partial t} = 0$$

for arbitrary σ, but if one accepts that the interaction of this field with the electromagnetic field is given by the normal prescription

$$\underset{\sim}{\nabla} \rightarrow \underset{\sim}{\nabla} - \frac{ie}{\hbar c}\underset{\sim}{A} \qquad \frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} + \frac{ie}{\hbar}A^0$$

then a value for σ of $e/\hbar c^2$ can be obtained, where e is the electronic charge.

Chapters 2 - 5 deal with the particle-like solutions of (1.1) and their stability. It is found that this field has serious defects and in an attempt to overcome some of these, a generalisation of this field is considered in chapters 6 - 9. In chapter 10 some suggestions for extending the theory are proposed.

2. PARTICLE-LIKE SOLUTIONS OF 1.6

2.1 Introduction

This equation has been investigated by several authors. There is a countable infinity of particle-like solutions, the simplest having no nodes, the next simplest one node, etc. The first (nodeless) solution has been obtained numerically and approximated variationally by Betts[10] et alii. Glasko, Leriust and Terletskii[10] have obtained the first five solutions numerically. The object of this chapter is to check these calculations, and to extend the number of known solutions.

For small r', one can expand φ' in a series solution

$$\varphi' = a_0 + a_2 r'^2 + a_4 r'^4 + a_6 r'^6$$

The terms in the power series a_2, a_4, a_6 . . . are determined in terms of a_0 in Appendix A. In general for an arbitrary a_0, φ' would not be particle-like but for large r would oscillate about the special solutions ± 1, being asymptotic to these solutions as $\frac{\sin(\sqrt{2}r' + \beta)}{r'}$ where β is some constant. For certain values of a_0, however, φ' does not oscillate about ± 1, but tends monotonically to zero with asymptotic form $\alpha e^{-r'}/r'$. The problem is to find those values of a_0 which give rise to this behaviour since it is only those solutions which are acceptable. We denote by A(k), the value of a_0 for which we get a particle-like solution having k nodes. Thus the problem is to find the A(k) for k = 0, 1, 2, . . . etc.

Alternatively, instead of starting with the trial value of a_0 and the series expansion, one can start with the asymptotic form of φ'

In/

In general an arbitrary value of d would not give a solution for which $\frac{d\varphi'}{dr'}$ was zero at $r' = 0$. But for certain values of d, denoted D(k), this condition would be met. The problem is to find the D(k), k = 0, 1, 2, . . .

Certain observations can be made.

(i) If φ' is a solution of (1.6), then so is $-\varphi'$. This means that one need only seek solutions for which A(k) is positive, or if preferable, D(k) is positive, though one can not simultaneously insist on having both A(k) and D(k) positive.

(ii) If $a_0 \gtrless A(k)$, then φ' oscillates about ∓ 1 respectively, (provided one takes A(k) positive), for k even and about ± 1 for k odd.

(iii) If $d \gtrless D(k)$, then $\left(\frac{d\varphi'}{dr'}\right)_{r'=0} \gtrless 0$ for $D(k) > 0$ for solutions with an even number of nodes i.e. k = 0, 2, 4, 6 and vice versa for k odd.

(iv) A particle-like solution φ' can be shown to satisfy certain integral relations for this field. These are

$$\int \varphi'^2 r'^2 dr' = \frac{1}{4}\int \varphi'^4 r'^2 dr' = \frac{1}{3}\int \left(\frac{d\varphi'}{dr'}\right)^2 r'^2 dr' \quad (2.1)$$

$$\int \varphi' r'^2 dr' = \int \varphi'^3 r'^2 dr'$$

These integral relationships can be used as a check on numeric accuracy by evaluating the integrals for a given numerical attempt at φ' and observing how closely the equations (2.1) are satisfied.

2.2 Numerical Calculations

In numerical (digital) solutions of differential equations, it

is/

is necessary to replace the derivatives of the function in terms of the values of the function at discrete points, and then to solve the system by solving the resulting finite difference equations. Such numerical methods do not restrict the direction of integration, i.e. one can (a) start at small r' and integrate to large r' (b) start at large r' and integrate to low r', (c) do both i.e. integrate out from small r' and integrate in from large r' to some common point r'_c, say. Solutions of (1.6) were attempted using each of these methods. Method (a) can provide a useful method of solving (1.6) when we make use of observation (ii). The method consists of taking an arbitrary value of a_o, using the series solution for φ' as a starting point and integrating (1.6) to large r' where the behaviour of φ' is examined. If φ' is oscillating about +1 for a_o = positive then the chosen value of a_o is too small. One increases a_o until a value is obtained for which φ' oscillates about -1 for $r' \to \infty$. This value of a_o is too large. One now has two values for a_o, one which is too small and one which is too large and these values sandwich A(k). The method then consists of contracting the upper and lower bounds on A(k) until the desired accuracy is reached. Method (b) is essentially the same as (a) but in the reverse direction. One uses the asymptotic form for φ' as a starting point and integrates (1.6) to $r' = 0$. The object is to find a value of d for which $\left(\frac{d\varphi'}{dr'}\right)_{r'=0}$ is negative and one for which this is positive, then one can contract these upper and lower/

lower bounds to D(k) until $\left(\frac{d\varphi'}{dr'}\right)_{r'=0}$ is acceptably zero.

Method (c) is a combined process. It consists of integrating outwards from $r' = 0$ using the series solution for φ' as a starting point with some trial value of a_o, to some value of r', say r'_c, and also integrating inwards to r'_c from large r' using the asymptotic form for φ' with some trial value of d. In general the values of φ' at r'_c obtained from integrating outwards and inwards would not be the same. The problem is to find those values of a_o and d for which φ' is matched at r'_c. (It can be shown that the solution is matched if both φ' and $\left(\frac{d\varphi'}{dr'}\right)$ are continuous at r'_c). The details of one method of systematically correcting a_o and d to satisfy the matching conditions are given in Appendix B.

The main method of solving the finite difference equations used to approximate (1.6) was the predictor-corrector method of Milne, though several other methods were tried. Apart from finding solutions to (1.6) directly, solutions were also found by solving some transformed versions of (1.6). The transformation $Z = r'\varphi'$ brings (1.6) to the form

$$\frac{d^2 Z}{dr'^2} = Z - \frac{Z^3}{r'^2} \qquad (2.2)$$

from which the first derivative term is absent. This equation was found superior to (1.6) in some respects and was often used.

For the higher order solutions, φ' becomes increasingly sharp around the origin. When integrating outwards on (1.6) or (2.2), it was necessary to use quite a few terms in the series expansion for φ' or/

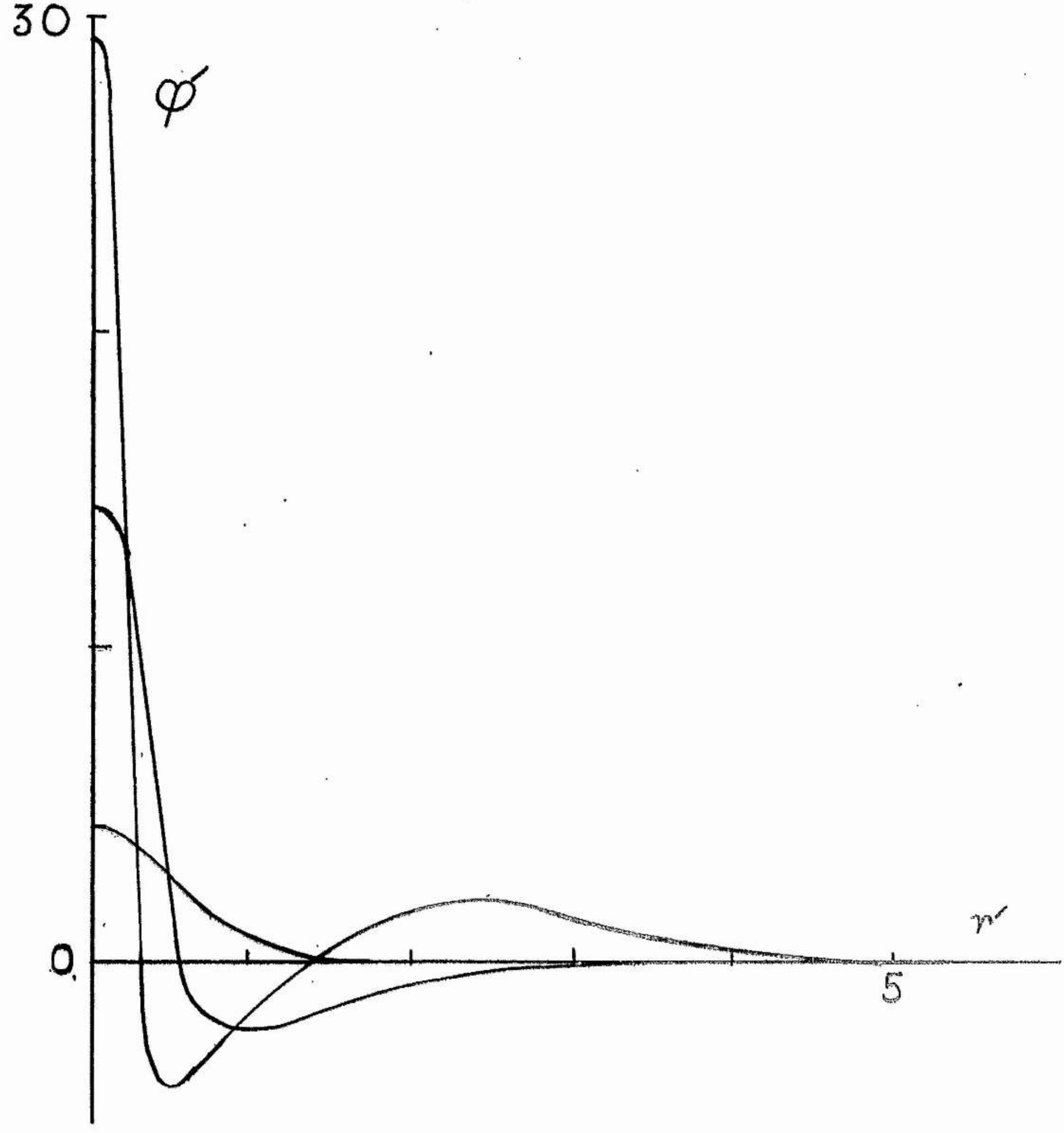
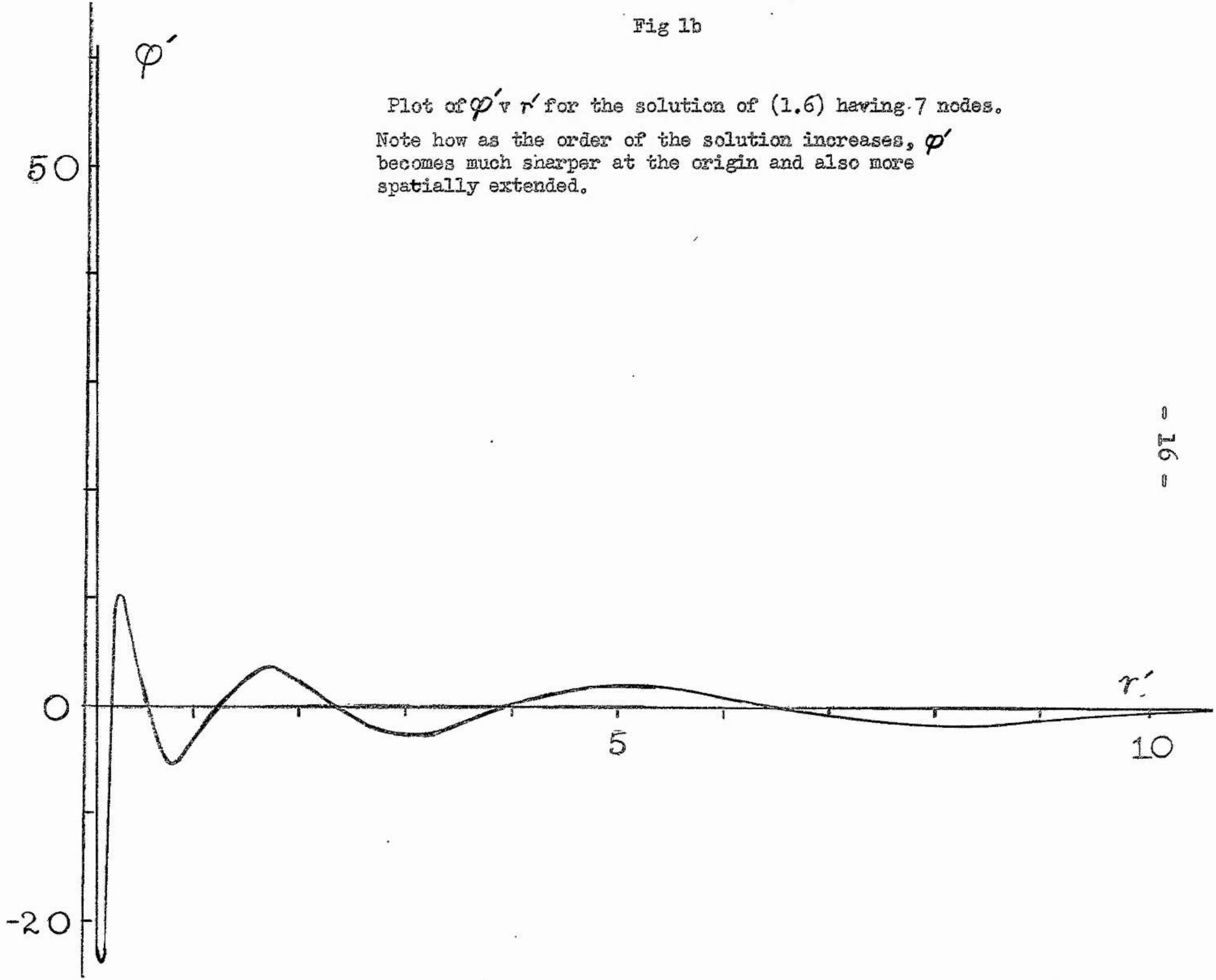


Fig 1a

Plot of φ' v r' for the three lowest-order solutions of (1.6).

Fig 1b



Plot of φ' v r' for the solution of (1.6) having 7 nodes.

Note how as the order of the solution increases, φ' becomes much sharper at the origin and also more spatially extended.

Table 1

The values of A(k), D(k) for the first eight solutions of (1.6). The values of the integrals for these solutions are also given, for comparison with (2.1)

k	A(k)	D(k)	$\int \varphi'^2 r'^2 dr'$	$\frac{1}{4}\int \varphi'^4 r'^2 dr'$	$\frac{1}{3}\int \left(\frac{d\varphi'}{dr'}\right)^2 r'^2 dr'$	$\int \varphi' r'^2 dr'$	$\int \varphi'^3 r'^2 dr'$
0	4.338 $\pm$ .001	2.712 $\pm$ .01	1.503	1.504	2.503	2.5	2.52
1	-14.045 $\pm$ .015	17.2 $\pm$ .1	9.6	9.5	9.5	11.0	11.1
2	29.045 $\pm$ .03	84 $\pm$ 1	29.1	28.8	28.8	27.6	27.3
3	-49.4 $\pm$ .1	379 $\pm$ 5	64.7	64.3	64.9	50.3	50.1
4	74.8 $\pm$ .3	1640 $\pm$ 20	120.9	120.4	120.7	84.4	84.2
5	-105.2 $\pm$ .7	6670 $\pm$ 70	205.	202	205	129	129
6	141.6 $\pm$ 1.5	27,200 $\pm$ 200	315.	316	318	175	176
7	-183.5 $\pm$ 2	112,600 $\pm$ 2000	460	464	470	229	233

or Z and also to use a small mesh size. (as small as .0001 for the eighth solution). Even so, equations (1.6) and (2.2) were satisfactory for integrating outwards. But if one wanted to integrate inwards using method (b), then these equations were found to be unsuitable because φ' was so sharp for $r' \sim 0$, that it was difficult to satisfy the condition $\frac{d\varphi'}{dr'} = 0$ at $r' = 0$. In an attempt to overcome this difficulty, (1.6) was transformed from an equation in (φ', r') to one in (φ', X) where $X = 1/r'$

This yields

$$\frac{d^2\varphi'}{dX^2} = \frac{\varphi' - \varphi'^3}{X^4} \qquad (2.3)$$

The transformation $X = 1/r'$ has the property of changing the critical region around r' zero to the region $X \rightarrow \infty$, and this was found to be useful in determining the values of D(k) for the higher node solutions.

The first three solutions are shown graphically in Fig. 1a. In Fig. 1b the eighth solution (7 nodes) is shown, indicating how sharp φ' becomes near the origin for the higher node solutions and also showing how it becomes more spatially extended. The values of A(k) and D(k) for the first eight solutions are given in Table 1 where the values of the integrals are also given. One can see that the integral relations (2.1) are very well obeyed for the lower order solutions, the discrepancy in the equalities being less than 1%. As the order of the solution increases the deviation from equality increases but even for the 8th solution, this deviation is still less than 3%. There seemed little point in continuing beyond the 8th solution so no more solutions were/

were found.

2.3 Analogue Calculations

An alternative method of solving (1.6) is to solve it not on a digital computer but on an analogue computer. The power of an analogue computer lies in the fact that it can continuously integrate and thus appears in principle well suited to solving differential equations since there is no need to approximate the derivatives of a function by finite differences.

There are however serious drawbacks in using an analogue computer, and this apparent superiority is not always justified. For equation (1.6) the nodeless solution was obtained by this method but the higher-order solutions could not be. A value of 4.6 for A(o) was obtained which compares reasonably with the numerically obtained value of 4.338.

2.4 Summary

Particle-like solutions of (1.6) having respectively 0, 1, . . . 7 nodes are obtained. A variety of numerical methods was used to obtain these solutions, the nodeless being obtained by analogue computation as well as by digital. A measure of the accuracy of the solutions can be obtained by observing how closely the integral relations (2.1) are satisfied for a given solution. Even for the eighth solution, the deviation/

deviation is less than 3% and for the lower solutions it is less than 1%. As we go to higher node solutions, φ' becomes increasingly sharp at the origin, as well as becoming more spatially extended.

3. STABILITY BY FIRST-ORDER PERTURBATION THEORY

3.1 Introduction

In this chapter the stability of the nodeless particle-like solution of (1.1) with time dependence of the form (1.3) is examined from the viewpoint of first-order perturbation theory. The time development of ψ is determined by the equation

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = K^2\psi - \mu^2\psi\psi^*\psi \qquad (1.1)$$

If we denote by subscript o the undisturbed state then $\psi_0 = \varphi_0\, e^{-i\omega t}$ where φ_0 is a solution of (1.4), is an exact solution of (1.1) for all time. But what happens if ψ_0 is disturbed by a small amount at some time. Does the disturbed state ψ remain in some sense close to ψ_0, or does it ultimately exhibit a different behaviour? It has been shown[6] that if $\omega = 0$, then ψ_0 is unstable when disturbed and ψ would not in general remain close to ψ_0. We do not know how ψ_0 behaves on being disturbed when ω is not zero, however, and this we now set out to determine.

Let us seek a disturbed solution of (1.1) of the form $\psi = \psi_0 + \psi_1$ where ψ_1 is assumed small, at least initially. If we expand (1.1) in terms of ψ_1, keeping only up to the first order in ψ_1, then we obtain the pair of equations

$$\nabla^2\psi_1 - \frac{1}{c^2}\frac{\partial^2\psi_1}{\partial t^2} = \left(K^2 - 2\mu^2\psi_0^*\psi_0\right)\psi_1 - \mu^2\psi_0^2\psi_1^* \qquad (3.1)$$

$$\nabla^2\psi_1^* - \frac{1}{c^2}\frac{\partial^2\psi_1^*}{\partial t^2} = \left(K^2 - 2\mu^2\psi_0^*\psi_0\right)\psi_1^* - \mu^2\psi_0^{*2}\psi_1$$

In analogy with the theory of the normal modes of vibration of a system about equilibrium we try a solution to (3.1) in the form

$$\psi_1 = \left[\eta(r)\, e^{-i\Omega t} + \chi^*(r)\, e^{+i\Omega^* t}\right] e^{-i\omega t} \tag{3.2}$$

where Ω is a complex constant. The general solution will then, hopefully, be a linear combination of such solutions over all allowed "normal frequencies" Ω.

If one inserts the form (3.2) into (3.1) and uses the transformations (1.5), one gets the pair of coupled second-order differential equations

$$\begin{aligned} \left[\nabla'^2 + \frac{(\Omega'+\omega')^2 - 1}{(1-\omega'^2)} + 2\varphi_0'^2\right]\eta &= -\varphi_0'^2\chi \\ \left[\nabla'^2 + \frac{(\Omega'-\omega')^2 - 1}{(1-\omega'^2)} + 2\varphi_0'^2\right]\chi &= -\varphi_0'^2\eta \end{aligned} \tag{3.3}$$

where Ω', ω' are defined by

$$\Omega' = \Omega_r' + i\,\Omega_i' = \Omega/KC \qquad\qquad \omega' = \omega/KC \tag{3.4}$$

In the analysis to follow φ_0' is taken to be the nodeless solution of (1.6). Equation (3.3) is an eigenvalue problem. For a given ω' we require to find the eigenvalues Ω', and eigenfunctions η, χ. We observe that if $(\eta, \chi, \Omega', \omega')$ is a solution of (3.3) then so are $(\chi, \eta, -\Omega', \omega')$, $(\eta^*, \chi^*, \Omega'^*, \omega')$, $(\eta, \chi, -\Omega', -\omega')$ solutions, and that there is no loss of generality in taking $\Omega_r' > 0$, $\Omega_i' > 0$, $\omega' > 0$. The parameter ω' is further required by (1.5) to be less than 1. If Ω' is not purely real then $\exp(-i\Omega t) = \exp(-i\Omega_r t) . \exp(\Omega_i t)$ is/

is an increasing function of t implying that the disturbance ψ_1 will build up with time and destroy the state ψ_0. Our condition for stability is thus that the only eigenvalues Ω' of (3.3) are purely real.

Numerically it is easier to deal, not with η, χ, but with $g = r'\eta$, $f = r'\chi$. When η, χ, are spherically symmetric (3.3) takes the form

$$\left[\frac{d^2}{dr'^2} + \frac{(\Omega'+\omega')^2 - 1}{(1-\omega'^2)} + 2\varphi_0'^2\right] g = -\varphi_0'^2 f \qquad (3.5)$$

$$\left[\frac{d^2}{dr'^2} + \frac{(\Omega'-\omega')^2 - 1}{(1-\omega'^2)} + 2\varphi_0'^2\right] f = -\varphi_0'^2 g$$

with the boundary conditions $g = f = 0$ at $r' = 0$ $g, f \to 0$ as $r' \to \infty$

For nonspherically symmetric solutions, (3.3) can be separated in spherical polar coordinates by assuming that the angular dependence of η, χ is given in terms of the spherical harmonics Y_ℓ^m

There is no coupling between different ℓ values and (3.3) can then be reduced to

$$\left[\frac{d^2}{dr'^2} - \frac{\ell(\ell+1)}{r'^2} + \frac{(\Omega'+\omega')^2 - 1}{(1-\omega'^2)} + 2\varphi_0'^2\right] g_\ell = -\varphi_0'^2 f_\ell \qquad (3.6)$$

$$\left[\frac{d^2}{dr'^2} - \frac{\ell(\ell+1)}{r'^2} + \frac{(\Omega'-\omega')^2 - 1}{(1-\omega'^2)} + 2\varphi_0'^2\right] f_\ell = -\varphi_0'^2 g_\ell$$

where $g_\ell = r'\eta_\ell$, $f_\ell = r'\chi_\ell$ and η_ℓ, χ_ℓ are the "radial" wave functions in the expansion of η, χ.

In/

In general one would expect Ω' to be a complex parameter and thus the eigensolutions of (3.5), (3.6) to be complex. But under certain conditions f and g may be taken real. This case will be considered in section 3.2, with the more general complex solutions discussed in section 3.3.

3.2 Real Solutions

Let us consider the special case in which the quantities

$$\gamma^2 = \frac{1-(\Omega'+\omega')^2}{(1-\omega'^2)} \qquad \delta^2 = \frac{1-(\Omega'-\omega')^2}{(1-\omega'^2)}$$

are both real. Restricting ourselves initially to spherically symmetric solutions, (3.5) then assumes the form

$$\begin{aligned}\left(\frac{d^2}{dr'^2} - \gamma^2 + 2\varphi_0'^2\right)g &= -\varphi_0'^2 f \\ \left(\frac{d^2}{dr'^2} - \delta^2 + 2\varphi_0'^2\right)f &= -\varphi_0'^2 g\end{aligned} \qquad (3.7)$$

where g and f can now be taken real.

There is a known solution to (3.7): $\Omega' = 0$, (i.e. $\gamma^2 = \delta^2 = 1$), $g = -f = r'\varphi_0'$ for any ω'. This however leads to the result $\psi_1 = 0$, and for that reason is considered trivial.

It is convenient to split the analysis of (3.7) into two parts (i) the case $\gamma^2 = \delta^2$ and (ii) the case $\gamma^2 \neq \delta^2$. Only in the former case are square integrable (γ^2, δ^2 both positive) solutions found to exist. In appendix D we consider the case of γ^2, δ^2 not both positive, when non square integrable solutions are found to exist.

3.2.1./

3.2.1. The Case $\gamma^2 = \delta^2$

Excluding the trivial case $\Omega' = 0$, $\gamma^2 = \delta^2 = 1$, real and equal γ^2 and δ^2 can occur only if $\omega' = 0$ and in addition Ω' is purely real or purely imaginary.

Adding and subtracting (3.7) with $\gamma^2 = \delta^2$ gives the two equations

$$\left[\frac{d^2}{dr'^2} - \gamma^2 + \varphi_o'^2\right] y = 0 \tag{3.8}$$

$$\left[\frac{d^2}{dr'^2} - \gamma^2 + 3\varphi_o'^2\right] z = 0 \tag{3.9}$$

where $y = (g - f)$ and $z = (g + f)$

Equation (3.8) is found to have only the trivial solution $y = r'\varphi_o'$ for $\gamma^2 = 1$. For large r', z has the asymptotic form $z = Pe^{-\gamma r'}$ (since (3.9) is homogeneous we are free to 'normalise' P to unity). Numerically (3.9) can be solved by taking a trial value of γ, and, using the asymptotic form for z with this value as starting point, integrating in to $r' = 0$. In general $(z)_{r' = 0}$ will not be zero for the chosen value of γ, and one must then correct γ in a systematic way until this condition is satisfied. A method of doing this is given in Appendix E . Solutions to (3.9) were sought numerically in this manner using for φ_o' a numerically obtained solution (see chapter 2). A solution for $\gamma^2 = 16.6$ was found corresponding to $\Omega' = 3.95i$, $\omega' = 0$, but no other solutions were obtained/

obtained. In appendix F we give the results of some other attempts at finding solutions to (3.8), (3.9).

3.2.2 Nonexistence of solutions when γ^2 and δ^2 are real and positive but unequal.

If γ^2 is not equal to δ^2 (i.e. $\omega' \neq 0$, $\Omega' \neq 0$) the possibility of finding eigenvalues which are purely imaginary does not exist, and the only possibility of getting solutions with γ^2, δ^2 real is if Ω' is purely real. In this subsection we seek solutions of this form, but find that none exist. Because $\Omega'_i = 0$, the values of γ^2, δ^2 permissible can be bounded. It is shown that solutions need be sought only for values of γ^2, δ^2 satisfying the inequality $\gamma^2 + \delta^2 \leqslant 2$:

$$\gamma^2 = \frac{1 - (\Omega' + \omega')^2}{(1 - \omega'^2)} \qquad \delta^2 = \frac{1 - (\Omega' - \omega')^2}{(1 - \omega'^2)}$$

$$\therefore \gamma^2 + \delta^2 = 2 - \frac{2\Omega'^2}{(1 - \omega'^2)} \leqslant 2 \qquad \text{since } \Omega' \text{ must here be real}$$

and $\omega'^2 < 1$. It is also necessary that γ^2, δ^2 be positive to give square integrable solutions.

Eigensolutions to (3.7) were sought numerically. The method consisted of generalising the approach used to solve (3.9). Equation (3.7) has an asymptotic form $g \rightarrow e^{-\gamma r'}$ $f \rightarrow Pe^{-\delta r'}$. Using this asymptotic/

asymptotic form, with trial values of γ, δ, P, (3.7) was integrated from large r' to r' = 0 where g, f were evaluated. Only if one has the 'correct' values of the parameters would g, f be simultaneously zero at r' = 0. The method then consists of correcting the parameters until this condition is met. No solutions for $\gamma^2 \neq \delta^2$ were found.

Instead of seeking solutions to (3.7) in terms of γ^2, δ^2 and from them evaluating Ω_r', ω', one might prefer to solve (3.7) directly in terms of Ω_r', ω'. In such an approach one wants a bound on Ω_r'. For Ω_r', ω' chosen positive, it is not difficult to show that Ω_r' must satisfy

$$\Omega_r' < (1 - \omega')$$

This gives Ω_r' a maximum upper bound of 1, and in general a value less than this. A search for eigenvalues of (3.7) in terms of (Ω_r', ω') was made. No solutions for $\omega' \neq 0$ were found.

3.2.3. Non-spherically Symmetric Solutions

In this section we are interested in solving (3.6) for $\ell = 1, 2, \ldots$ when g_ℓ, f_ℓ are real. Equation (3.6) has one known solution:

$$\ell = 1 \qquad g_1 = f_1 = r'\left(\frac{d\varphi'}{dr'}\right) \qquad \Omega' = 0$$

corresponding to the solution $\eta = \chi = \left(\frac{d\varphi'}{dr'}\right) Y_1^m$. This is a solution of (3.1) as a consequence of the translational invariance of (1.1). In Appendix G it is shown that for $\omega' = 0$, (3.6) can have no solutions for/

for any $\ell > 1$.

When $\omega' \neq 0$, solutions to (3.6) were sought using the methods of 3.2.2, but none were found.

3.3 Solutions for Complex Ω'

When Ω' is complex and $\omega' \neq 0$, (3.5) can not be reduced to an essentially real problem and solutions must be sought to (3.5) directly. The method of attack is similar to that given in section 3.2. For large r', (3.5) has asymptotic form of solution

$$
\begin{aligned}
g_r &= \left[X \cos(k_g r') + E \sin(k_g r') \right] e^{-\Sigma r'} \\
g_i &= \left[X \sin(k_g r') - E \cos(k_g r') \right] e^{-\Sigma r'} \\
f_r &= \left[P \cos(k_f r') + Q \sin(k_f r') \right] e^{-\Sigma' r'} \\
f_i &= \left[P \sin(k_f r') - Q \cos(k_f r') \right] e^{-\Sigma' r'}
\end{aligned}
\qquad (3.10)
$$

where $g = g_r + ig_i$, $f = f_r + if_i$, Σ, Σ' are defined by

$$
\Sigma = \sqrt{\frac{1 + \Omega_i'^2 - (\Omega_r' + \omega')^2 + \sqrt{\left[1 + \Omega_i'^2 - (\Omega_r' + \omega')^2\right]^2 + 4\,\Omega_i'^2 (\Omega_r' + \omega')^2}}{2(1 - \omega'^2)}}
\qquad (3.11)
$$

$$\Sigma' = \sqrt{\frac{1+\Omega_i'^2-(\Omega_r'-\omega')^2+\sqrt{\left[1+\Omega_i'^2-(\Omega_r'-\omega')^2\right]^2+4\Omega_i'^2(\Omega_r'-\omega')^2}}{2(1-\omega'^2)}}$$

and k_g and k_f are given by

$$k_g = \frac{\Omega_i'(\Omega_r'+\omega')}{(1-\omega'^2)\,\Sigma} \qquad k_f = \frac{\Omega_i'(\Omega_r'-\omega')}{(1-\omega'^2)\,\Sigma'}$$

The set of equations (3.5), (3.10) is homogeneous and we are thus free to normalise one of X, E, P, Q. Since the known solution $\Omega_r' = 0$, $\Omega_i' = 3.95$, $\omega' = 0$ corresponds to a nonzero X and one can use this as a starting point for examining the $\omega' \neq 0$ region, it is convenient to choose X as the parameter to be normalised. As (3.5), (3.10) stand, there are five parameters Ω_r', Ω_i', E, P, Q, but only four boundary conditions $g_r = g_i = f_r = f_i = 0$ at $r' = 0$. A Newton-type correction method would not be useful for this problem since we have more parameters to be adjusted than there are equations to correct them. However, a method which is capable of correcting as many parameters as we choose is the minimisation method, the details of which are given in Appendix E.

Before giving the results of the analysis it is desirable to bound the eigenspace of Ω' and this is now done.

3.3.1 Bounds on the eigenvalues Ω_r', Ω_i'

It/

It has been pointed out that there is a lower bound to Ω_r', Ω_i' of zero since there is no loss in generality in considering Ω_r', Ω_i' to be positive. In this section we derive a severe upper bound. Equation (3.5) can be recast in a form examined by Barston[12]:

$$\left(\omega_B^2 I - 2i\omega_B A - H\right)\xi = 0 \tag{3.12}$$

where ω_B, H, iA, ξ are defined by

$$\omega_B = \frac{\Omega'}{(1-\omega'^2)^{\frac{1}{2}}} \quad ; \quad H = \begin{bmatrix} -\frac{d^2}{dr'^2} + 1 - 3\varphi_0'^2 & 0 \\ 0 & -\frac{d^2}{dr'^2} + 1 - \varphi_0'^2 \end{bmatrix}$$

$$iA = \frac{-\omega'}{(1-\omega'^2)^{\frac{1}{2}}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \xi = \begin{pmatrix} z \\ y \end{pmatrix} = \begin{pmatrix} g+f \\ g-f \end{pmatrix} \tag{3.13}$$

Barston shows (3.12) that this leads to the inequality for ω_B.

$$|\omega_B|^2 \leqslant -\text{(lowest eigenvalue of H)}$$

provided Ω_i' is nonzero. Now we have effectively determined the lowest eigenvalue of H when we solved (3.8), (3.9). The lowest eigenvalue of H is the lowest eigenvalue of the upper diagonal element of H, namely -15.6 (The lowest eigenvalue of H corresponds to the greatest eigenvalue Ω_i' of (3.9) which can be denoted Ω_i' (0), since it is the value of Ω_i' at $\omega' = 0$)

Thus

$$\Omega_r'^2 + \Omega_i'^2 \leqslant 3.95^2\,(1 - \omega'^2) \tag{3.14}$$

Further, again assuming $\Omega_i' \neq 0$,

$$\left[\text{Real}\,\omega_B\right]^2 = \frac{(\xi, iA\xi)^2}{(\xi,\xi)^2} \leqslant \frac{(\xi,\xi)(iA\xi, iA\xi)}{(\xi,\xi)^2} = \frac{\omega'^2}{(1-\omega'^2)}$$

(3.15)

Equations (3.14) and (3.15) then allow one to bound the eigenspace of Ω_r', Ω_i' as

$$\Omega_i' \leqslant 3\cdot 95\,(1-\omega'^2)^{\frac{1}{2}}$$

$$\begin{cases} \Omega_r' \leqslant 3\cdot 95\,(1-\omega'^2)^{\frac{1}{2}} \\ \Omega_r' \leqslant \omega' \end{cases} \qquad (3.16)$$

provided $\Omega_i' \neq 0$.

3.3.2 Values of Ω' obtained by solving (3.5)

Initially solutions were sought for specified ω' i.e. a value of ω' was chosen and the values of Ω_r', Ω_i', E, P, Q corrected by the minimisation routine until a solution was found. Notationally, this can be expressed as minimising

$F\,(\omega';\ \Omega_r', \Omega_i', E, P, Q)$, a positive quantity which vanishes if and only if a solution of (3.5) has been attained (See Appendix E). The notation indicates that parameters to the left of the semicolon are held fixed while those to the right are optimised by the minimisation routine. For a given ω', a unique eigenvalue was found by this process. Always Ω_r' was zero, i.e. Ω' was purely imaginary. In table 2, Ω_i' is tabulated against ω'. In fig. 2 this information is displayed graphically. Although/

Table 2. Values of Ω_i' calculated from numerical and variational methods

ω'	Ω_i' (Numerical)	Ω_i' (Variational)
0	3.95	3.96
.1	3.93	3.94
.2	3.85	3.86
.3	3.73	3.74
.4	3.54	3.56
.5	3.29	3.32
.6	2.97	2.99
.7	2.51	2.57
.8	1.89	2.00
.9	.92	1.18

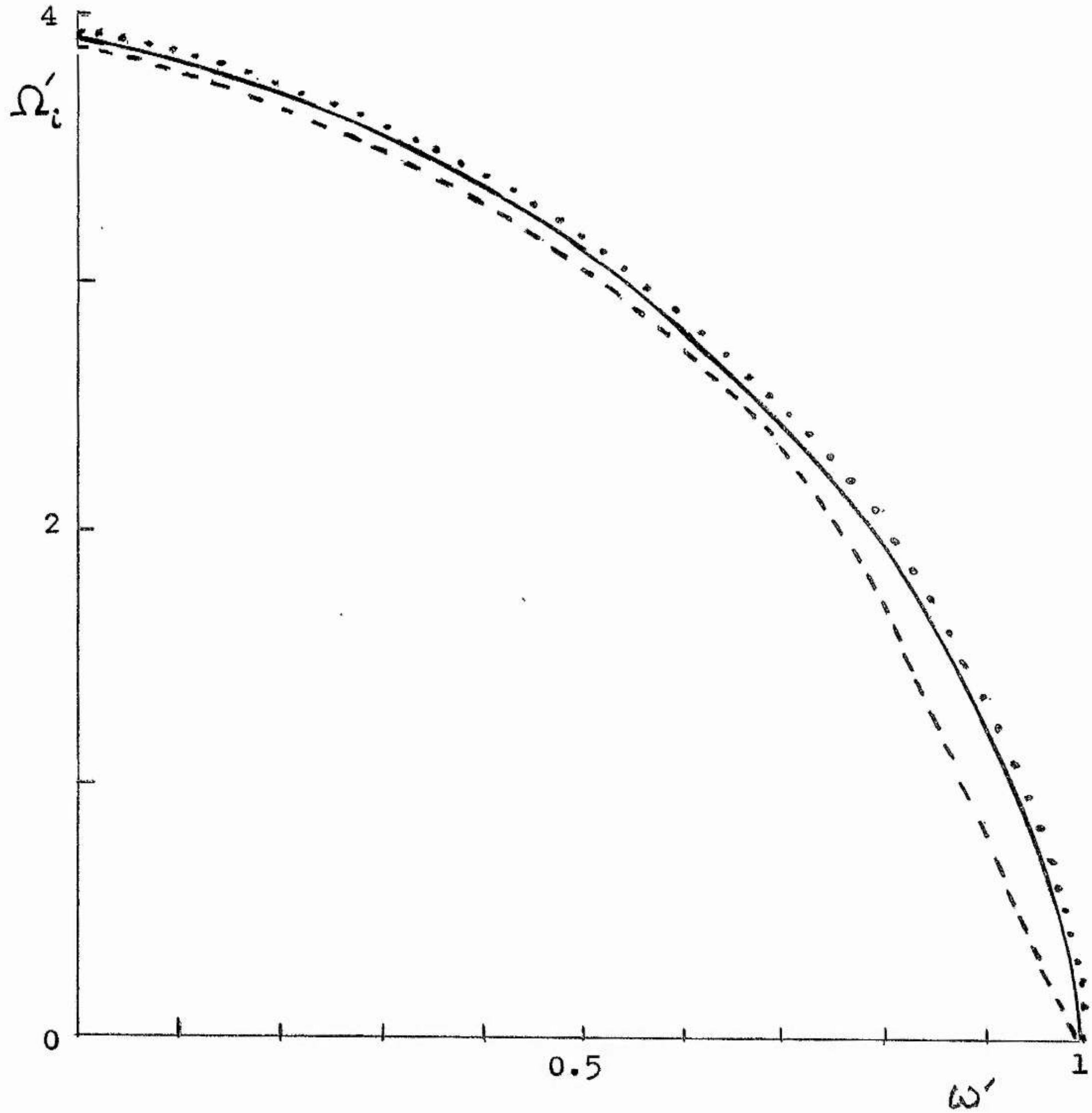


Fig 2

Eigenvalues of (3.5) The eigenvalue Ω_i is shown plotted against ω'. The dashed curve indicates results obtained by numerical methods while the solid curve shows results obtained by the variational method. The dotted curve limits the region within which the eigenvalues must lie.

Although for a given ω', a unique value of Ω_i' was found, several values of E, P, Q were obtained. In appendix H it is shown that the lack of uniqueness in E, P, Q is a consequence of Ω_r' being zero.

Solutions for complex Ω' were also sought by minimising

$$F(\omega', \Omega_r' ; \Omega_i', E, P, Q)$$

$$F(\Omega_r' ; \omega', \Omega_i', E, P, Q)$$

$$F(\Omega_i' ; \omega', \Omega_r', E, P, Q)$$

$$F(; \omega', \Omega_i', \Omega_r', E, P, Q)$$

No solutions other than those already found were obtained.

It has been pointed out, that in (3.10) one could arbitrarily normalise one of X, E, P, Q. X was chosen and normalised to unity. Now this implies that any solution must have a nonzero Cos term in its asymptotic form for g. Perhaps solutions exist to this problem with a vanishing Cos term in their asymptotic expansion. Solutions of this form were sought. Other normalisation conditions were also tried: each of E, P, Q were normalised in turn. No new solutions were found.

Fig. 2. thus shows all the known eigenvalues to (3.5). For a given ω', there is only one nonzero eigenvalue Ω', and it is purely imaginary. Also drawn in fig. 2 is Barston's limiting curve $\bar{\Omega}_i = 3.95\,(1 - \omega'^2)^{\frac{1}{2}}$. Notice how good a bound this is to the numerically obtained values for Ω_i'. It might be thought that the bounding curve and/

and the eigenvalue curve should actually be identical, and that the difference between them is a consequence of some numerical inaccuracy. In Appendix I it is shown that this is not so.

3.3.3 Non spherically symmetric solutions

When η, χ are not spherically symmetric, the problem is to find complex solutions to (3.6). It is shown in Appendix J that (3.6) can have no complex eigenvalues for $\ell > 1$.

3.4 Variational Calculations

It is easy to get a variational method to solve this type of problem when $\omega' = 0$, or when Ω' = pure real, but to obtain a method for Ω' complex is more difficult. A method is here proposed.

Let us define an operator π by

$$\pi = (\omega_B^2 - 2iA\omega_B - H) \tag{3.17}$$

where ω_B, iA, H are as defined by (3.13). Both iA and H are Hermitian operators, but since ω_B can be complex, π is not. By doubling the dimensions of the vector space we can construct the Hermitian operator

$$L = \begin{pmatrix} 0 & \pi \\ \pi^{\dagger} & 0 \end{pmatrix} \tag{3.18}$$

and so sensibly seek the real eigenvalues of L i.e. solutions to the problem

$$L\bar{\xi} = \lambda\bar{\xi} \tag{3.19}$$

where/

where λ is a real eigenvalue and $\bar{\xi} = \begin{pmatrix} u \\ v \end{pmatrix}$ where u, v are 2-component vectors. Because iA, H are real as well as Hermitian $\pi^{\dagger} = \pi^{*}$ and thus there is no loss in generality in taking $v = u^{*}$. Equation (3.19) can then be simplified to

$$\begin{pmatrix} 0 & \pi \\ \pi^{*} & 0 \end{pmatrix} \begin{pmatrix} u^{*} \\ u \end{pmatrix} = \lambda \begin{pmatrix} u^{*} \\ u \end{pmatrix} \qquad (3.20)$$

yielding the variational principle $\delta[\lambda] = 0$ for arbitrary δu where $[\lambda]$ is defined by

$$[\lambda] = \frac{\int (u, u^{*}) \begin{pmatrix} 0 & \pi \\ \pi^{*} & 0 \end{pmatrix} \begin{pmatrix} u^{*} \\ u \end{pmatrix}}{\int (u, u^{*}) \begin{pmatrix} u^{*} \\ u \end{pmatrix}} \qquad (3.21)$$

$$\doteq \frac{\mathrm{Re} \int (u \pi u)}{\int u u^{*}} \qquad (3.22)$$

$[\lambda]$ is a function of ω_B, ω_B^{*} and the problem is to find those values of ω_B for which λ vanishes.

For φ_0' we use the simple variational approximation $4\sqrt{2}\, e^{-\sqrt{3}r'}$. Recall that for the case $\omega' = 0$, using this approximation for φ_0' with a trial wave function $r' e^{-ar'}$, a value for Ω' of 3.96 was obtained which compares favourably with numerically obtained value of 3.95. (See Appendix F). It can be shown that for the $\omega' = 0$ case this variational principle is equivalent to the one in Appendix F and thus the approximation for φ_0' should be satisfactory for the present purposes. As a trial wave function we take

$$u = \begin{pmatrix} \gamma r' e^{-\alpha r'} \\ \beta r' e^{-\alpha r'} \end{pmatrix} \tag{3.23}$$

which is an obvious generalisation of the wave function used in Appendix F . α, β, γ are complex parameters $\alpha = \alpha_r + i\alpha_i$ etc. Variation of $[\lambda]$ with respect to the parameters yields a set of non linear simultaneous equations which can be solved by standard minimisation methods. A discussion of some of the results obtained from this variation principle follows.

3.4.1 Results of variational calculations

Equation (3.19) is homogeneous and so one of γ_r, γ_i, β_r, β_i can be normalised. Initially γ_r was normalised and $[\lambda]$ varied with respect to the other parameters γ_i, β_r, β_i, α_r, α_i. We are thus interested in solving the simultaneous equations

$$[\lambda] = \frac{\partial[\lambda]}{\partial\gamma_i} = \frac{\partial[\lambda]}{\partial\beta_r} = \frac{\partial[\lambda]}{\partial\beta_i} = \frac{\partial[\lambda]}{\partial\alpha_r} = \frac{\partial[\lambda]}{\partial\alpha_i} = 0 \tag{3.24}$$

This can be done by constructing the function F.

$$F = [\lambda]^2 + \left(\frac{\partial[\lambda]}{\partial\gamma_i}\right)^2 + \left(\frac{\partial[\lambda]}{\partial\beta_r}\right)^2 + \left(\frac{\partial[\lambda]}{\partial\beta_i}\right)^2 + \left(\frac{\partial[\lambda]}{\partial\alpha_r}\right)^2 + \left(\frac{\partial[\lambda]}{\partial\alpha_i}\right)^2$$

and minimising it to zero. Solutions were sought by minimising F subject to the following three alternative conditions:

(i) ω' fixed ; Ω^0_r , Ω^0_i , γ_i, β_r, β_i, α_r, α_i variable

(ii)/

(ii) Ω_r' fixed ; ω', Ω_i', γ_i, β_r, β_i, α_r, α_i variable.

(iii) all 8 parameters variable

No solutions for which Ω_r' was non zero were found. For a given ω' a unique value of Ω_i' was found. This is given in Table 2 and plotted as a function of ω' in fig. 2. Note the good agreement with the numerically obtained values.

In the above γ_r was normalised to unity. The whole procedure was repeated for β_i normalised to unity. No non-trivial solutions were found. The results of this variational approach thus agree very well with the results obtained by solving (3.5) numerically: for a given ω' a unique value of Ω' is found which is always purely imaginary.

3.5 Summary

In this chapter a method of analysing the stability of time-dependent particle-like solutions to small disturbances is given. Application of first-order perturbation theory reduces the problem to solving an eigenvalue problem for a coupled set of linear second-order differential equations. If the eigenvalue Ω' has a non zero imaginary part Ω_i' then the particle is unstable.

For any given ω', it is found that a unique value of Ω' exists, but this value is purely imaginary. Despite intensive searching using both numerical and variational techniques no eigenvalue for which the real part Ω_r' is non zero has been found for this field. Nonspherically symmetric solutions are also sought but no non trivial solutions occur.

Because/

Because for any ω' there exists an eigenvalue Ω' with non zero imaginary part, it is concluded that the lowest-order particle-like solution for this field is unstable to small disturbances whether it is time-dependent or not.

4. STABILITY BY DIRECT PERTURBATION METHODS

4.1 Introduction

In chapter 3, the stability of the lowest-order (nodeless) particle-like solution with time dependence of the form (1.3) was examined from the point of view of first-order perturbation theory. This technique gives a useful indication of what might happen to such a solution when it is disturbed. The results of the analysis imply that it is unstable irrespective of whether it is time-dependent or not.

In chapter 3 we expand (1.1) in terms of $\psi_1 = \psi - \psi_0$ keeping only up to first-order terms in the perturbation ψ_1. In this section, the stability of ψ_0 is examined when all orders in ψ_1 are kept. One would expect first-order perturbation theory to give good results for the case of stable disturbances, for then ψ_1 would not only start small but would remain small for all time. For disturbances to which ψ_0 was not stable the results of first-order perturbation theory should not be so good, for instability would manifest itself by giving a complex value to Ω' which implies that ψ_1 would increase steadily with time and that there would come a time when ψ_1^2 would not be negligible compared with ψ_0 and thus that the assumption that one could discard terms in ψ_1 of order higher than the first would not be valid. In such an event one would still expect first-order perturbation theory to give a measure of the decay time but not to reveal the whole truth.

The stability of ψ_0 is now examined by direct perturbation methods: ψ_0 and $\frac{\partial \psi_0}{\partial t}$ are given disturbances at a given time and then the time development/

development of the disturbed state determined by solving (1.1).

If we define $\underset{\sim}{\rho} = K\underset{\sim}{r}$; $\tau = Kct$; $\Psi = \frac{\mu}{K}\psi$ then (1.1) can be reduced to the apparently parameterless form

$$\nabla_{\rho}^{2}\Psi - \frac{\partial^{2}\Psi}{\partial\tau^{2}} = \Psi - \Psi\Psi^{*}\Psi \tag{4.1}$$

where $\Psi = \Psi_0 + \Psi_1$. The undisturbed state Ψ_0 is given by

$$\Psi_0 = \frac{\mu}{K}\psi_0 = \frac{\mu}{K}\varphi_0 e^{-i\omega t} = \sqrt{1-\omega'^2}\,\varphi_0' e^{-i\omega'\tau} \tag{4.2}$$

For this analysis, φ_0' is taken to be the lowest-order (nodeless) solution of (1.6). The solution of (1.6) is given as a function of r', whereas for (4.1) it is required as a function of ρ. However ρ and r' are related by $\rho = Kr = (1 - \omega'^2)^{-\frac{1}{2}} r'$.

The reduced energy density $\mathcal{E}' = \mathcal{E}\mu^2/K^4$ can be found from (1.8) to be

$$\mathcal{E}' = \left|\frac{\partial\Psi}{\partial\tau}\right|^2 + \left|\nabla_{\rho}\Psi\right|^2 + \left|\Psi\right|^2 - \tfrac{1}{2}\left|\Psi\right|^4 \tag{4.3}$$

Initially the stability of the time-independent ($\omega' = 0$) state ψ_0 will be examined. Only spherically symmetric disturbances are considered.

4.2 Stability of the nodeless time-independent particle-like solution

Let us consider initially the case $\omega' = 0$. Then for this case (4.2) reduces to

$$\Psi_0 = \varphi_0' \qquad \text{and} \qquad \rho = r'$$

Initially/

Initially a disturbance Ψ_1 to which Ψ_0 should be stable is considered. Later other types of disturbances are examined.

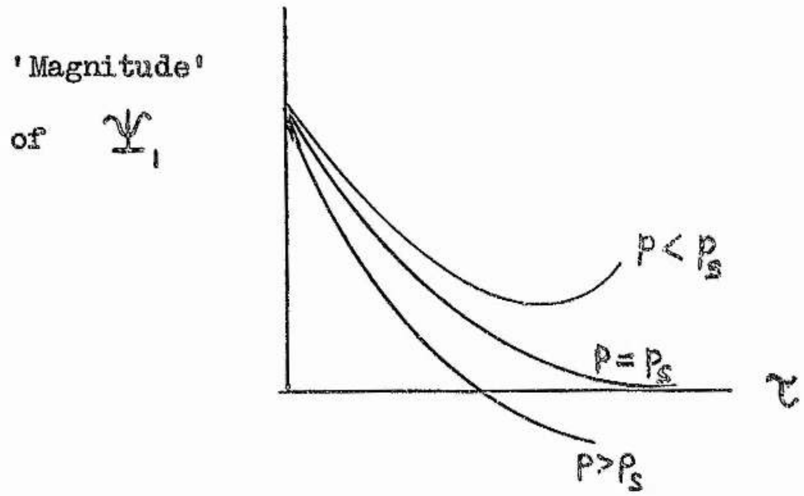
From the results of first-order perturbation theory the state Ψ_0 should be stable to the disturbance $\Psi_1 = \frac{Z(r')\, e^{-\Omega_i' \tau}}{r'}$ where z(r') is the solution of (3.9) and Ω_i' (= 3.95) is the corresponding eigenvalue. For increasing τ, Ψ_1 decays exponentially with time and $\Psi \to \Psi_0$. However numerically it will not be possible to produce this disturbance exactly, and since $\Psi_1 = \frac{Z(r')\, e^{+\Omega_i' \tau}}{r'}$ is also a solution of (3.9), any numeric attempt at the falling solution will couple in some of the rising term which will eventually dominate and destroy the state Ψ_0. Nevertheless it is possible to partially verify the results, since if one starts with the falling disturbance it should continue to follow an exponential decay for some time before the rising term makes its influence felt. To test this, the disturbance

$$\Psi_1\Big|_{\tau=0} = \frac{Z_e(e)\, e^{-p\tau}}{e}\Big|_{\tau=0} \qquad \frac{\partial \Psi_1}{\partial \tau}\Big|_{\tau=0} = -\frac{p Z_e(e) e^{-p\tau}}{e}\Big|_{\tau=0} \quad (4.4)$$

was applied, where $Z_e(e)$ is the solution of (3.9) appropriately transformed and p is some constant. The results of chapter 3 lead one to expect that for Ψ_0 to be stable to (4.4), p must have a value of the order of 4. [Ω_i'=3.95 was the value obtained by solving (3.9)].

The disturbance (4.4) was applied and some interesting results were obtained. It was found, that there was a value of p that made Ψ_0 essentially stable to the disturbance (4.4) (though eventually Ψ_1 did build up as expected). Let us denote this value of p by p_s. It was/

was also discovered that for $p > p_s$, $p < p_s$ the disturbance (4.4) always induced the same type of behaviour. Below, the 'magnitude' of Ψ_1 is drawn figuratively as a function of τ for various values of p.



For $p > p_s$, Ψ_1 always builds up in such a manner as to cause Ψ_0 to decay by what we shall call the dissipative mode, while for $p < p_s$ the decay assumes quite a different form, which we shall designate the singular mode.

In figs. 3 and 4 the time development of Ψ is shown for the cases p = 4.5 and p = 4. Ψ is plotted against ρ for various values of τ. The value of p_s obtained was 4.1 ± .05 to be compared with the value obtained from first-order perturbation theory of 3.95. Fig. 3 illustrates an example of the dissipative mode. There Ψ oscillates with steadily decreasing amplitude. Fig. 4 shows a singular decay. It is clear how Ψ rapidly builds up and goes singular at $\rho = 0$ in a very short time.

Other variations of disturbance (4.4) were tried. The expectation that/

that $\Psi_1 = \frac{Z_e(\rho)\, e^{-p\tau}}{\rho}$ be a disturbance to which Ψ_0 is stable arose from consideration of first-order perturbations. Equation (3.9), which essentially defines Ψ_1, is a homogeneous equation. Thus one should expect $a\Psi_1$ to be a satisfactory stable disturbance for arbitrary a provided a is not so large as to render invalid the assumption that Ψ_1 is a small disturbance.

Numerically it was found that when we applied the disturbance defined by (4.4), with Ψ_1 replaced by $a\Psi_1$ the behaviour as defined by (4.1) was virtually independent of a though there was some slight variation in the values of p_s recorded e.g. for $a = -1$, the value of p_s was $4.05 \pm .05$.

While it is possible to examine the stability of Ψ_0 by plotting Ψ as a function of ρ for various values of τ it is better to adopt a new method of representation. In chapter 1 it was pointed out that a physically sensible way of representing a particle in this theory was in terms of the energy density. We will now discuss the stability of a particle-like solution by examining the time evolution of the energy density. The undisturbed nodeless time-independent particle-like solution has a reduced energy density $\mathcal{E}'$ given by (4.3) and shown graphically in fig. 5. The size of the particle represented by this solution could be taken as the distance R marked in fig. 5. This is not a rigorously defined quantity but for the value of R chosen, one can see that $\mathcal{E}'$ is essentially different from zero only for $r' < R$, and that $\mathcal{E}'$ is effectively zero for all $r' > R$. Examination of fig. 1a shows for this value of R (i.e. 2 units of r'), that φ' is also essentially zero for/

for $r' > R$.

In figs. 6 and 7 we show the effects of the disturbances defined by (4.4) for $p = 3.5$ and $p = 5$, respectively, by plotting $\mathcal{E}'$ as a function of ρ for a sequence of values of τ. In fig. 7, one sees how the shape of the energy density changes with time. For increasing τ, $\mathcal{E}'$ tends to the zero value as the energy is pushed further and further away from the origin. This form of decay is denoted the dissipative mode, since the particle dissipates its energy when disturbed. An example of the singular mode is given in fig. 6. For increasing τ, $\mathcal{E}'$ becomes rapidly more negative in the region around the origin, finally developing a singularity there. Hence the name singular decay.

When disturbances of the form (4.4) were applied to Ψ_0, two and only two types of decay resulted viz. the dissipative and the singular. For disturbances not of the form (4.4), still only the same two decay modes were found. Some other types of disturbance are now considered.

$$\text{Disturbances} \quad \Psi_1\Big|_{\tau=0} \quad \text{and} \quad \frac{\partial \Psi_1}{\partial \tau}\Big|_{\tau=0} \tag{4.5}$$

were generated by a random number generator in the ranges -0.02 to 0.02 and -0.4 to 0.4 respectively and the time development of $\mathcal{E}'$ observed. Again only singular and dissipative decays were recorded. Fig. 8 shows a typical dissipative decay and fig. 9 a typical singular decay. In the singular decay the particle draws more and more negative energy into a decreasing region round the origin, compensating for this by increasing the amount of energy in the positive region. This decay is not a physically acceptable one and goes against the cherished postulate of nonlinear/

nonlinear field theory that there shall be no singularities. This decay is a consequence of the field permitting $\mathcal{E}'$ to be negative in certain regions of space. The dissipative decay seems an acceptable form of decay.. The particle simply radiates all its energy.

Because there is no catastrophic event in the decay of Ψ there is no obvious method of measuring the decay time absolutely. There is no obvious time at which one can say the 'particle' no longer exists but suggested methods for at least estimating the decay times of the state Ψ_0 are given. Examination of fig. 8 shows that after 3 units of τ, the energy density is not visibly different from zero anywhere, i.e. there is no region of space in which a localisation of the energy occurs. It is thus reasonable to take this as some measure of the decay time of Ψ_0 for the dissipative mode, but it is stressed that this definition is to some extent arbitrary. The decay time for the singular mode can be more 'rigorously' defined. An examination of fig. 9 shows that after approximately 1.5 units of τ, the energy density is becoming rapidly more singular. This can be more clearly seen from fig. 10 where $\mathcal{E}'_{min}$ the minimum value of $\mathcal{E}'$, is plotted against τ. Then it is clear that when $\tau \sim 1.5$, $\mathcal{E}'_{min}$ is definitely going singular and hence a value for the decay time of ~ 1.5 units of τ would be reasonable for this case.

Different random disturbances give slightly different decay times. A series of random disturbances were applied to Ψ_0 and the decay mode together with the decay time for that mode were recorded. An illustration is given in Table 3. From this it can be seen that there is some variation/

variation in the values of the decay times recorded but in general the values lie close to those obtained above. For the singular decay mode, lifetimes lie within the range $1 \rightarrow 2.5$ τ- units while for the dissipative mode in the range $2 \rightarrow 4$ τ - units.

Disturbances resembling the collision of the particle with some sort of projectile were tried. Fig. 11 shows such a form of disturbance. The disturbance given by

$$\Psi_1\Big|_{\tau=0} = 0 \qquad \frac{\partial \Psi_1}{\partial \tau}\Big|_{\tau=0} = 0 \qquad e < 3.5 \tag{4.6}$$

$$\Psi_1 = \frac{2.5\times 10^8 \times (e-3.5)\, e^{-5e}}{e} \qquad \frac{\partial \Psi_1}{\partial \tau}\Big|_{\tau=0} = \frac{1.0\times 10^8 \times (e-3.5)\, e^{-5e}}{e} \qquad e > 3.5$$

was applied. The reduced energy density $\mathcal{E}'$ immediately after the disturbance is applied is shown in fig. 11 a. (The reduced energy density immediately before the disturbance is applied i.e. for the undisturbed state, is shown in fig. 5). Initially the disturbance is small and well outside the particle radius R. (Although the disturbance is visibly small it contains a non negligible amount of energy since it lies well away from the origin). The situation .5 τ-units later is shown in fig. 11b. Frames b, c, d show the disturbance moving in on the particle with oscillations forming and travelling in. Outgoing oscillations do also occur but are very much smaller and are not visible on this scale. So far there has been no disturbance within the particle radius R and the 'particle' is as yet unaware of its impending fate. At e the disturbance is just beginning to enter the particle radius and the amplitude of the oscillations is increasing. The remaining three frames/

frames f, g, h show the violent interaction of the disturbance with the particle. Decay by the singular mode occurs within 0.2 τ- units after fig. 11h (using the definition of singular decay given previously).

In this type of interaction, where the disturbance is applied a long way from the particle, one should not start measuring the decay time until the disturbance has started entering the particle-radius, since the particle is essentially unaware of the disturbance until this time. Adopting this convention, the value of the decay time for the above disturbance is ~ 1.7 τ- units. Dissipative decays of the above form do also occur, but one finds in general, using disturbances of this type, that the probability of inducing a singular decay is greater than the probability of precipitating a dissipative decay.

So far little mention has been made of the charge Q. It is a fact of nature that Q is quantised in lumps of e i.e. $Q = 0, \pm e$ Now the time-independent state represents a neutral particle as can be seen from inserting the time dependence of the form (1.3) into the standard field theory definition for charge density given by (1.9). This gives

$$\rho = \frac{2 e \omega' K^3 (1 - \omega'^2)}{\hbar c \mu^2} \varphi'^2 \qquad (4.7)$$

which is zero only when ω' is zero. Thus, for the time-independent case ($\omega' = 0$), ψ_0 represents a neutral particle. But in general when one disturbs ψ_0 to ψ one alters the charge of the state from zero to some finite value. Now this goes against the laws of nature. Perhaps one/

one should try to assert charge quantisation and only consider disturbances for which Q remains zero. Disturbances satisfying this condition were tried but it was found that keeping Q zero did not make any difference to the decay times of ψ_0 . As the theory stands, charge does not appear to play any fundamental role.

4.3 Results for the time-dependent case $\omega' \neq 0$

So far the discussion has centered on the stability of the time-independent case ($\omega' = 0$). This proves a useful starting point for considering the time-dependent case ($\omega' \neq 0$). In fact, the behaviour of particle-like solutions in the time-independent and time-dependent cases is similar, but when the unperturbed particle-like solution is time-dependent it takes longer to decay.

As for the $\omega' = 0$ case, only two types of decay occur: singular and dissipative. In general ψ decays more slowly as ω' increases but this is not surprising since ψ also becomes more spatially extended as ω' increases. Figs. 12 and 13 show singular and dissipative decays for the case $\omega' = 0.8$, which we take as representative of time-dependent particle-like solutions. The disturbances are applied by a random number generator similar to that used for the $\omega' = 0$ case. Fig. 12 shows a singular decay. The behaviour in this case bears a close resemblence to the case $\omega' = 0$ (fig. 9), but the decay time is clearly longer. In fig. 14 a graph of $\mathcal{E}'_{min}$ v τ is drawn for this decay and we can see that $\mathcal{E}'$ goes singular for a value of $\tau \sim 2.8$, compared with/

with 1.5 for the $\omega' = 0$ case (fig. 10). The dissipative decay shown in fig. 13 is clearly similar to the dissipative decays for the $\omega' = 0$ case, (Compare with fig 8 say) but it is not until $\tau = 7$ that $\mathcal{E}'$ is not visibly different from zero in this case compared with $\tau = 3$ for the decay of fig. 8. A series of random disturbances for the case $\omega' = 0.8$ produces the spread of decay times given in Table 3. Comparison with the $\omega' = 0$ case shows that while the type of behaviour does not change, the decay times are longer for the $\omega' = 0.8$ case.

In figs. 15a and 15b we plot the reciprocal of the mean lifetime v ω' for the dissipative and the singular decay modes respectively. These curves have the same general shape as the graph of Ω_i' v ω' shown in fig. 2. Recall that $(\Omega_i')^{-1}$ gives a measure of the decay time of a state. The lower the value Ω_i' the longer the state takes to decay. First-order perturbation theory, however, does not suggest that there are two decay modes with different lifetimes, but does give results in qualitative agreement with direct perturbation methods.

4.4 Summary

The stability of the lowest-order particle-like solution is examined by direct perturbation on a digital computer, initially when the unperturbed particle-like solution is time-independent and then when it is time-dependent. It is found that for a wide variety of perturbations applied, the particle was always unstable decaying by either the singular or the dissipative decay mode. In the singular mode, the particle draws an/

an ever increasing amount of negative energy into a small region around the origin whereas in the dissipative mode it radiates all its energy to infinity.

As ω' increases it is found that the particle takes longer to decay, in qualitative agreement with the results of chapter 3.

Table 3

Table of decay modes and decay times for 15 random disturbances for the cases $\omega' = 0$, $\omega' = 0.8$.

$\omega' = 0$		$\omega' = 0.8$	
Decay mode	Decay time	Decay mode	Decay time
Singular	2.4	Singular	2.5
Dissipative	2.5	Singular	2.1
Singular	1.6	Dissipative	6.1
Dissipative	3.1	Singular	2.9
Singular	1.4	Singular	3.0
Singular	1.7	Singular	2.8
Singular	1.5	Dissipative	8.2
Dissipative	2.6	Dissipative	7.0
Singular	1.4	Singular	2.6
Singular	1.5	Dissipative	6.3
Dissipative	2.9	Singular	2.2
Singular	1.3	Singular	3.1
Dissipative	3.5	Dissipative	6.1
Singular	1.9	Singular	2.7
Dissipative	3.3	Singular	2.5

For the $\omega' = 0$ case the mean singular decay time is ~ 1.7 τ- units while for the $\omega' = 0.8$ case it is ~ 2.6 τ-units with standard deviations about the mean values of .3 and .4 τ- units respectively. The mean dissipative time for the case $\omega' = 0$ is 3 τ-units and for the case $\omega' = 0.8$ it is 6.7. The standard deviations in this case are respectively .4 and .8 τ- units.

From these calculations one can see that

(i) the dissipative decay is generally slower than the singular decay

(ii) the decay times for the $\omega' = 0.8$ case are longer than for the $\omega' = 0$ case.

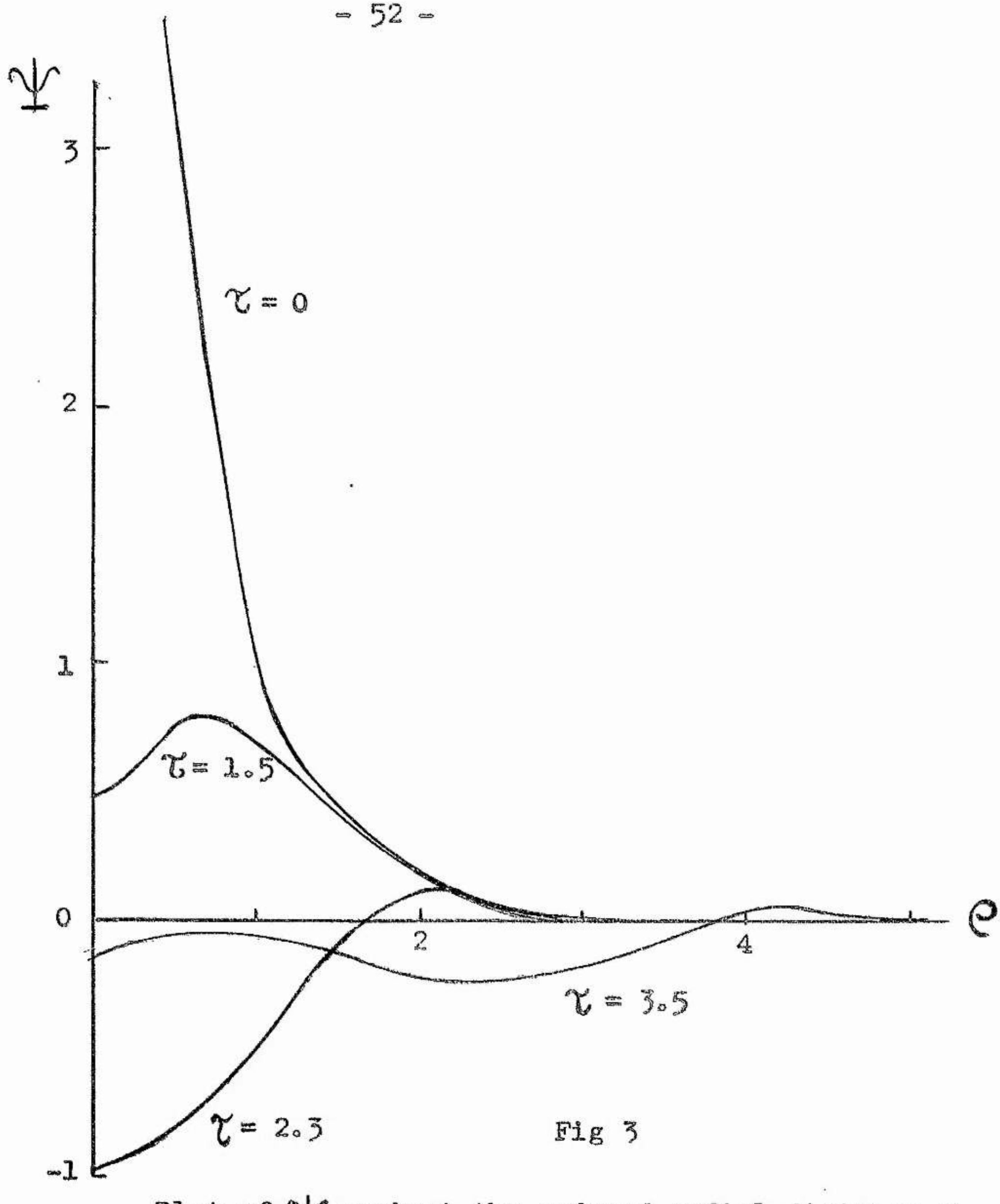
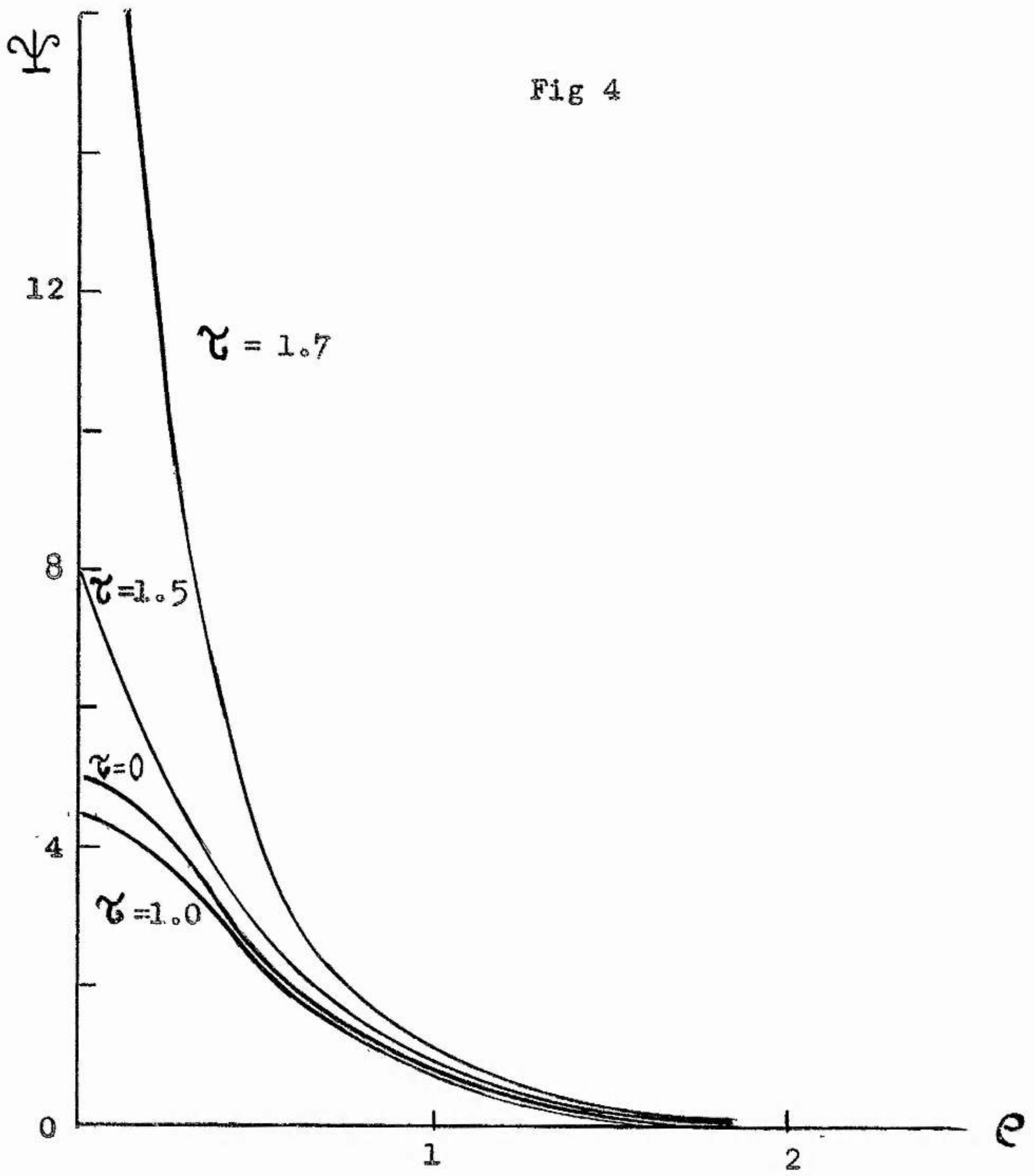


Fig 3

Plot of Ψ against the reduced radial distance ρ for a series of values of the reduced time τ. The disturbance given by (4.4) for p=4.5 induces a dissipative decay. Ψ oscillates in time with decreasing amplitude.



Plot of Ψ against the reduced radial distance ρ for a series of values of the reduced time τ. The disturbance, similar to that of fig 3 but for p=4, induces a singular decay. The amplitude of Ψ initially decreases but then rapidly builds up.

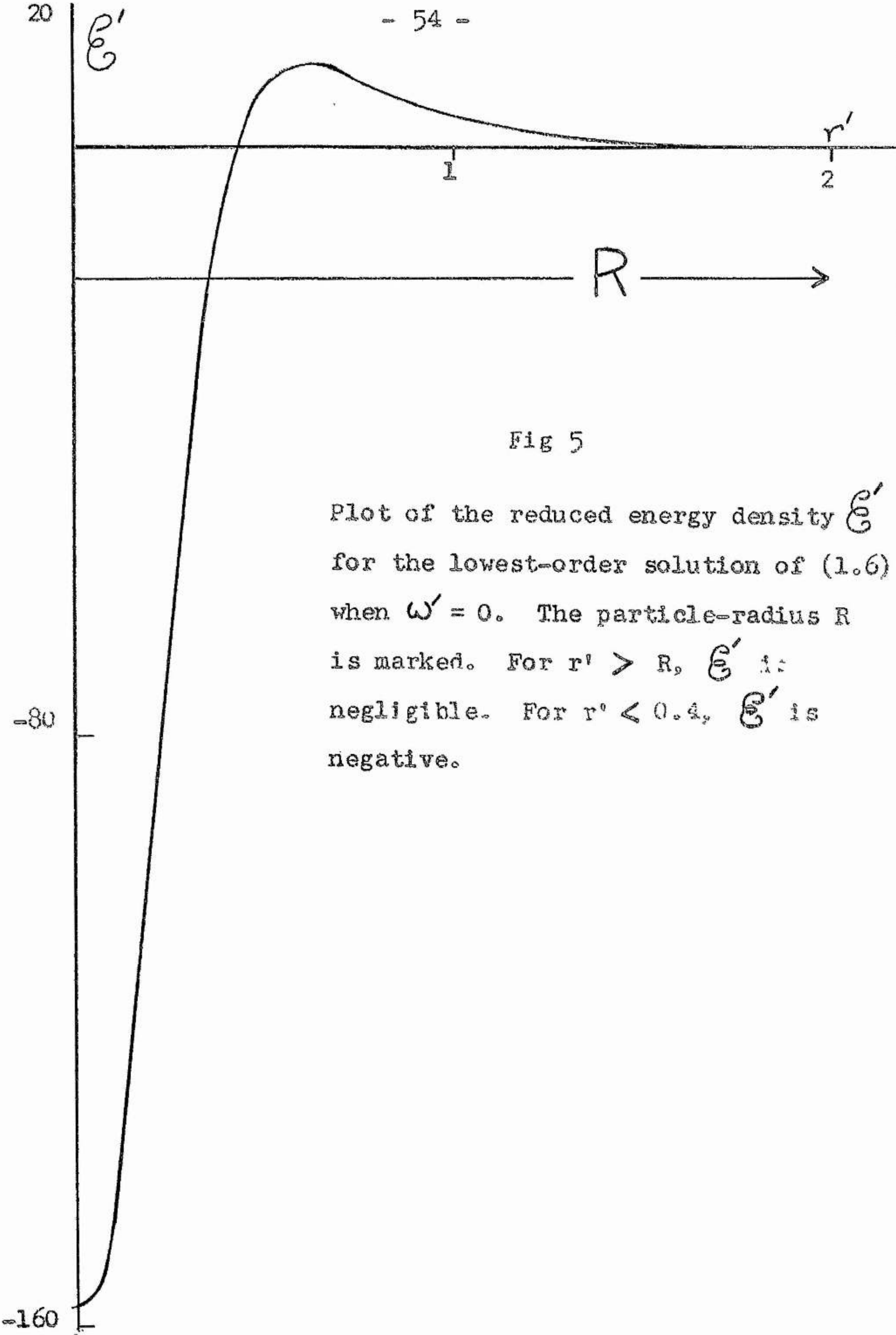


Fig 5

Plot of the reduced energy density $\mathcal{E}'$ for the lowest-order solution of (1.6) when $\omega' = 0$. The particle-radius R is marked. For $r' > R$, $\mathcal{E}'$ is negligible. For $r' < 0.4$, $\mathcal{E}'$ is negative.

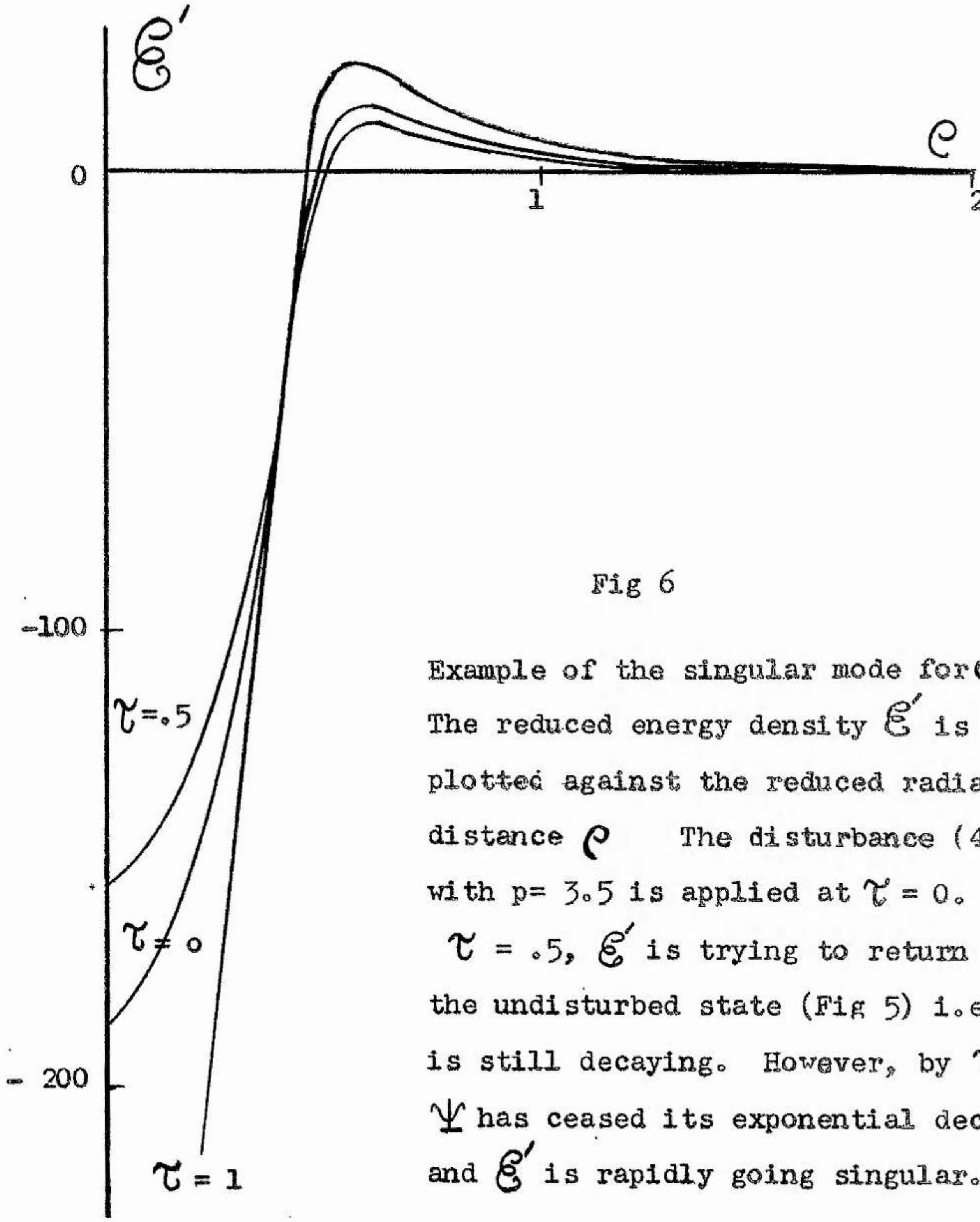


Fig 6

Example of the singular mode for $\omega = 0$. The reduced energy density $\mathcal{E}'$ is plotted against the reduced radial distance ρ The disturbance (4.4) with p= 3.5 is applied at $\tau = 0$. At $\tau = .5$, $\mathcal{E}'$ is trying to return to the undisturbed state (Fig 5) i.e. Ψ_1 is still decaying. However, by $\tau = 1$, Ψ has ceased its exponential decay and $\mathcal{E}'$ is rapidly going singular.

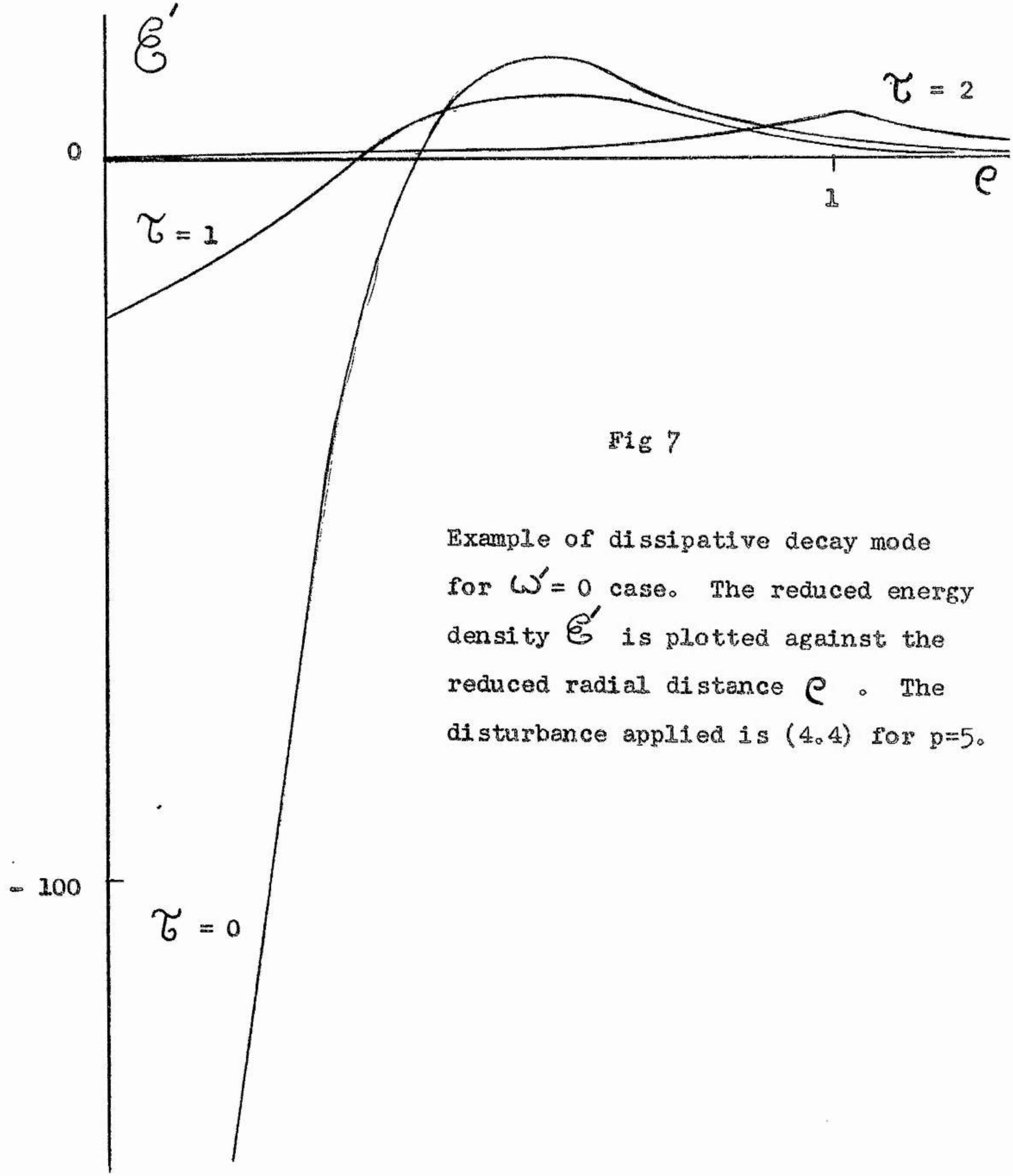


Fig 7

Example of dissipative decay mode for $\omega' = 0$ case. The reduced energy density $\mathcal{E}'$ is plotted against the reduced radial distance ρ. The disturbance applied is (4.4) for p=5.

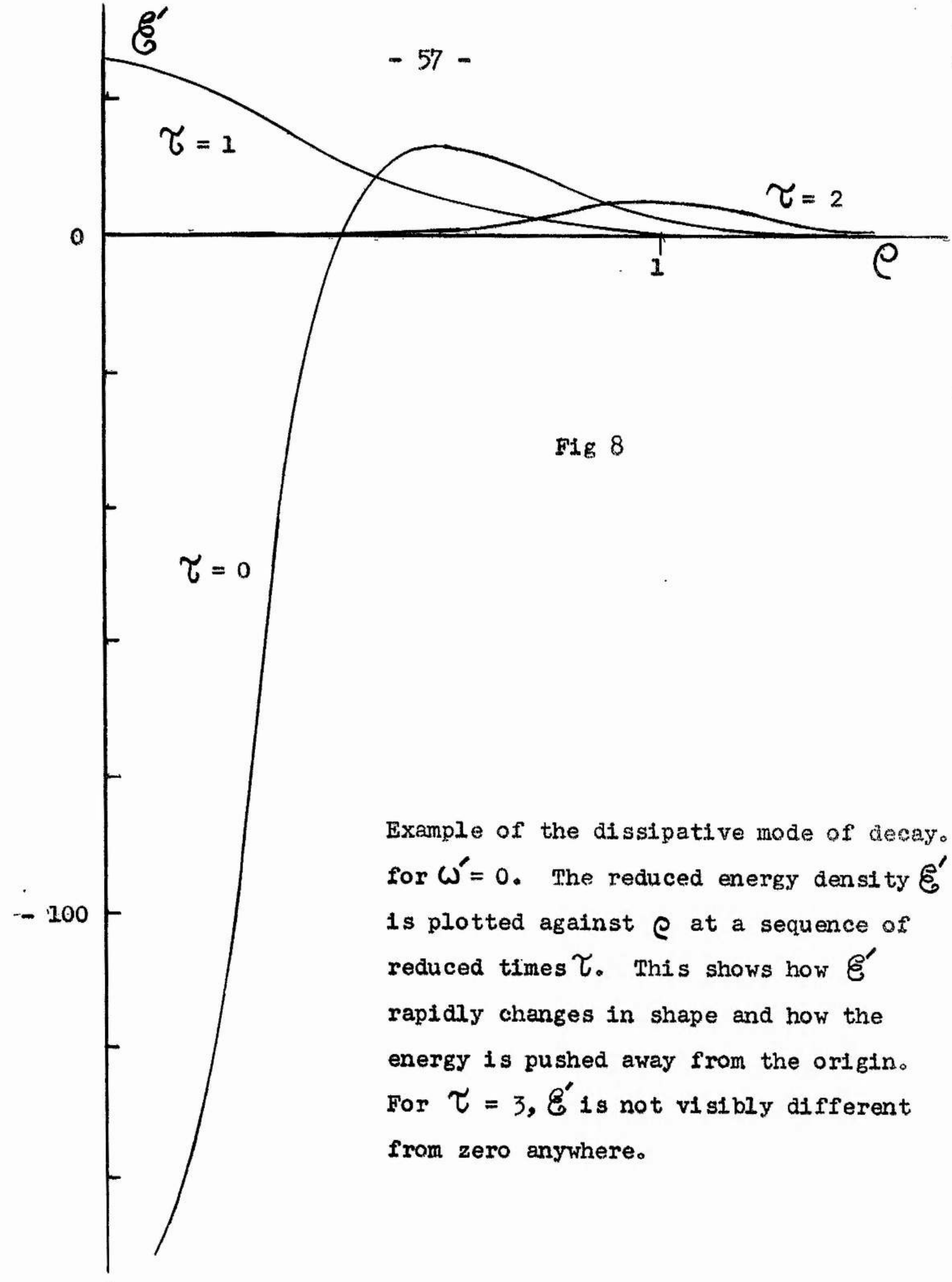


Fig 8

Example of the dissipative mode of decay. for $\omega' = 0$. The reduced energy density $\mathcal{E}'$ is plotted against ρ at a sequence of reduced times τ. This shows how $\mathcal{E}'$ rapidly changes in shape and how the energy is pushed away from the origin. For $\tau = 3$, $\mathcal{E}'$ is not visibly different from zero anywhere.

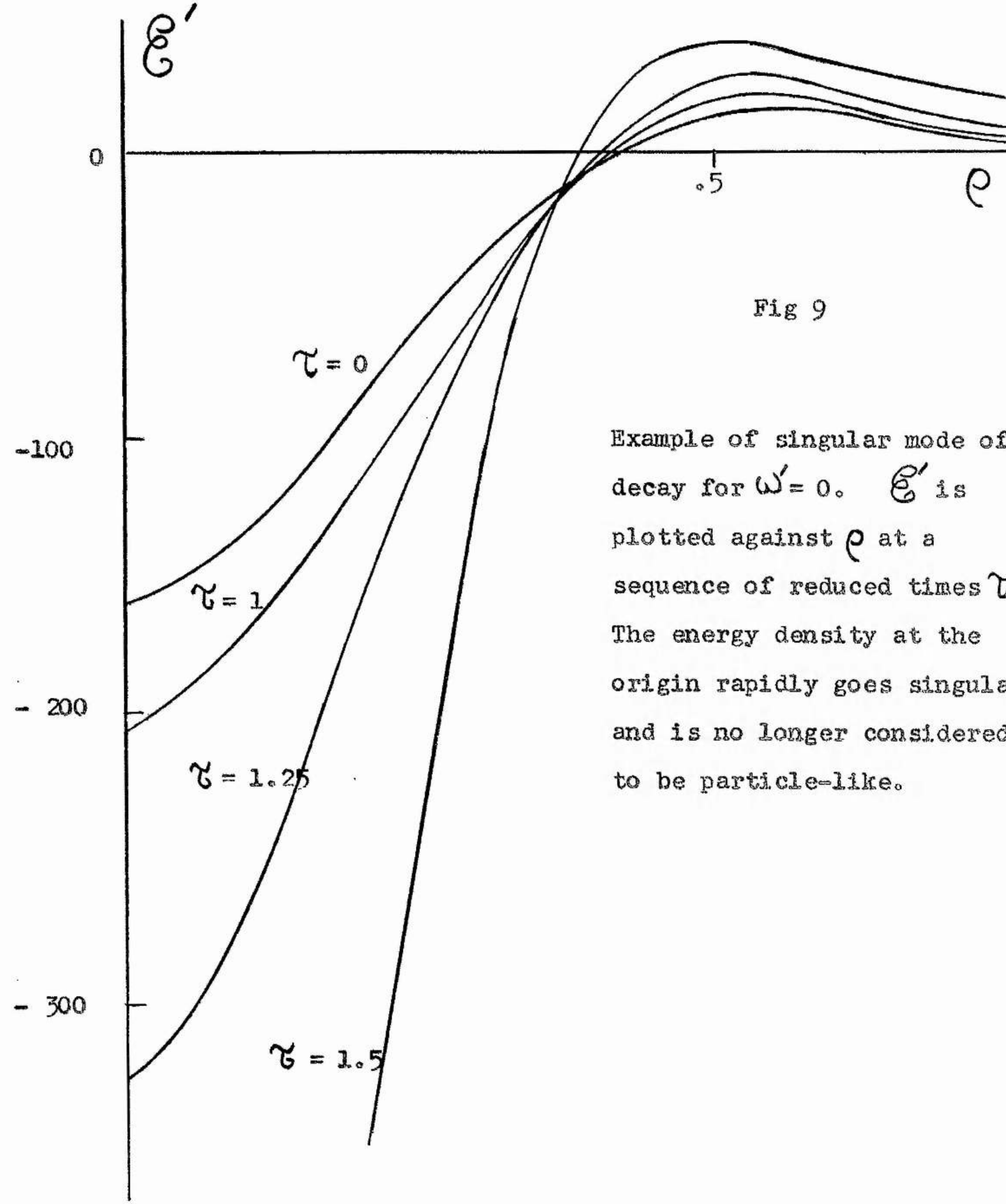


Fig 9

Example of singular mode of decay for $\omega' = 0$. $\mathcal{E}'$ is plotted against ρ at a sequence of reduced times τ. The energy density at the origin rapidly goes singular and is no longer considered to be particle-like.

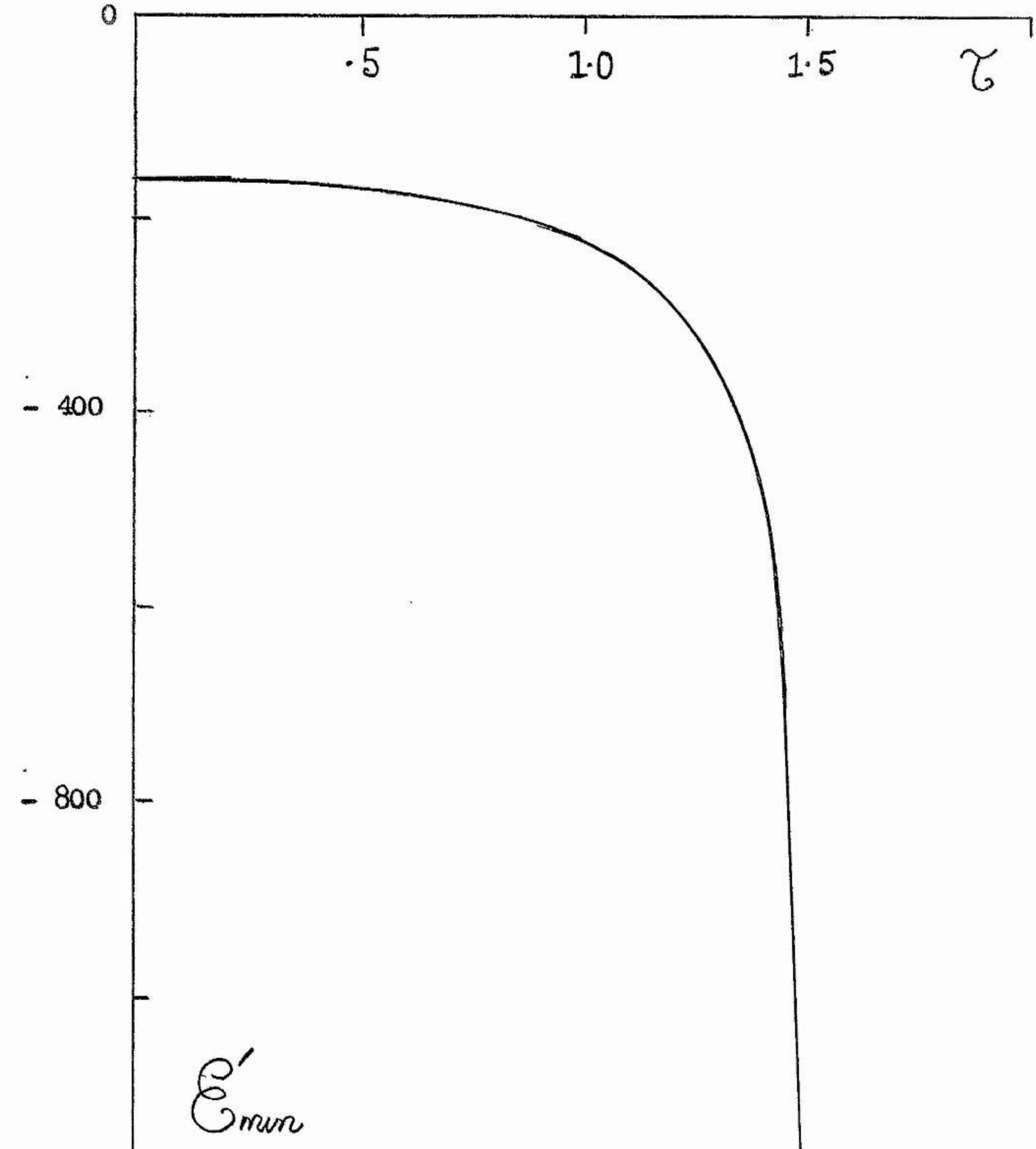


Fig 10

Plot of $\mathcal{E}'_{min}$ v τ for the disturbance shown in fig 9, showing how rapidly $\mathcal{E}'$ goes large and negative at $\tau \sim 1.5$

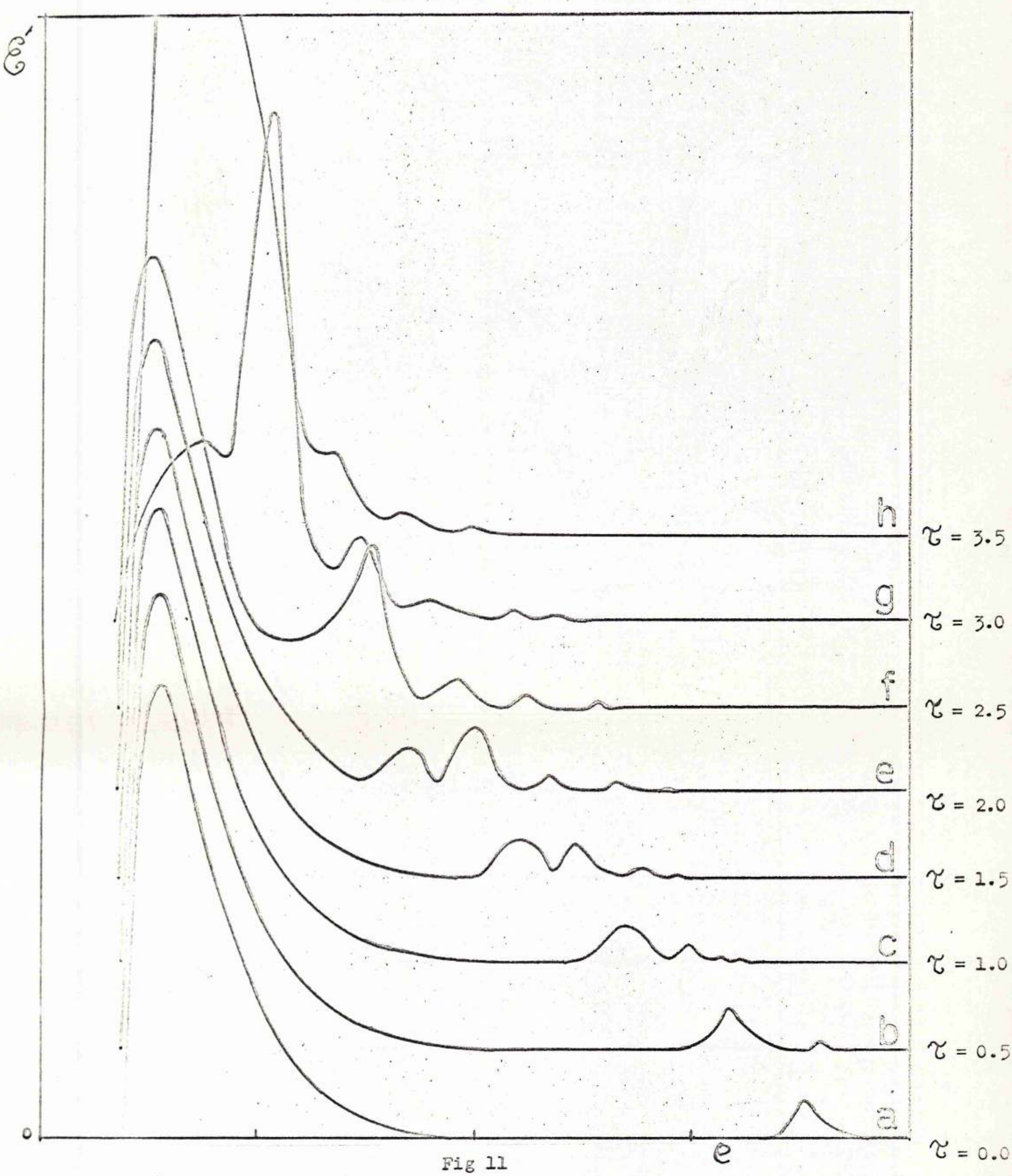


Fig 11

Plot of $\mathcal{E}'$ v ρ at a sequence of reduced times τ, the disturbance being applied at $\tau = 0.0$ (11a) whence it travels inwards. The origin of $\mathcal{E}'$ is displaced vertically for each time level for illustration purposes and for a similar reason the region where $\mathcal{E}'$ is negative is not plotted.

Scale:- $\mathcal{E}'$: $0 \rightarrow 11$ ρ : $0 \rightarrow 4$

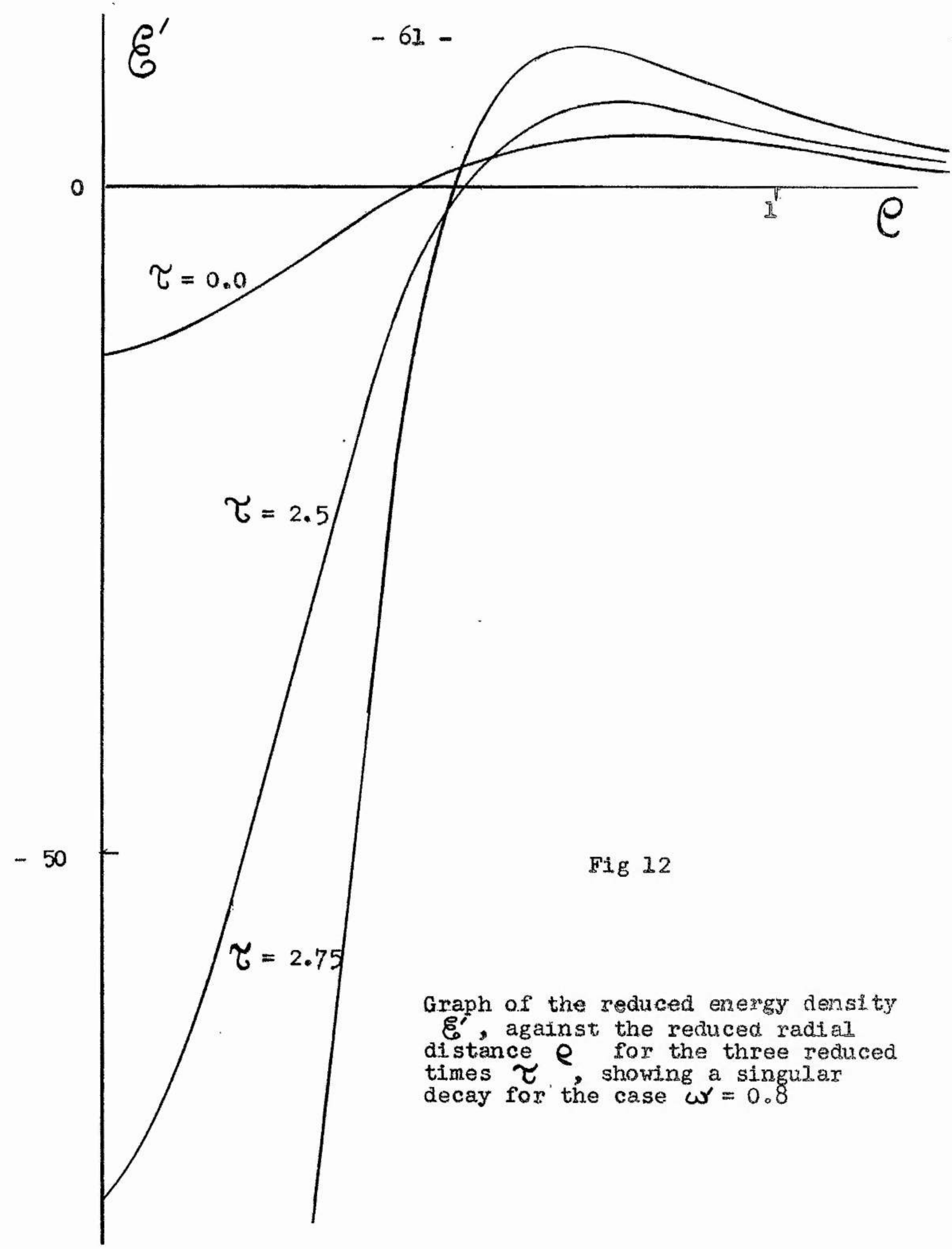


Fig 12

Graph of the reduced energy density $\mathcal{E}'$, against the reduced radial distance ρ for the three reduced times τ, showing a singular decay for the case $\omega = 0.8$

Fig 13

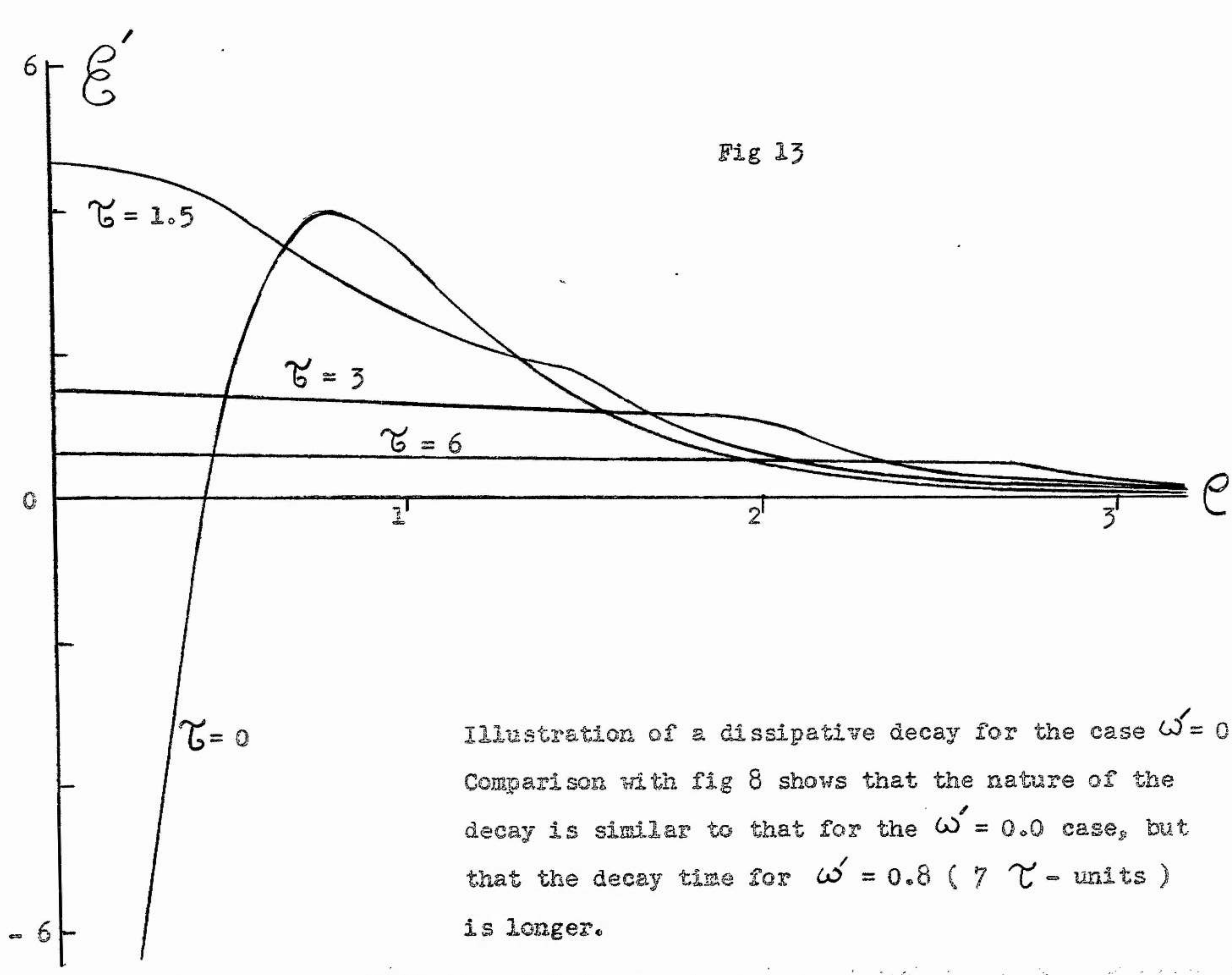


Illustration of a dissipative decay for the case $\omega' = 0.8$
Comparison with fig 8 shows that the nature of the decay is similar to that for the $\omega' = 0.0$ case, but that the decay time for $\omega' = 0.8$ (7 τ - units) is longer.

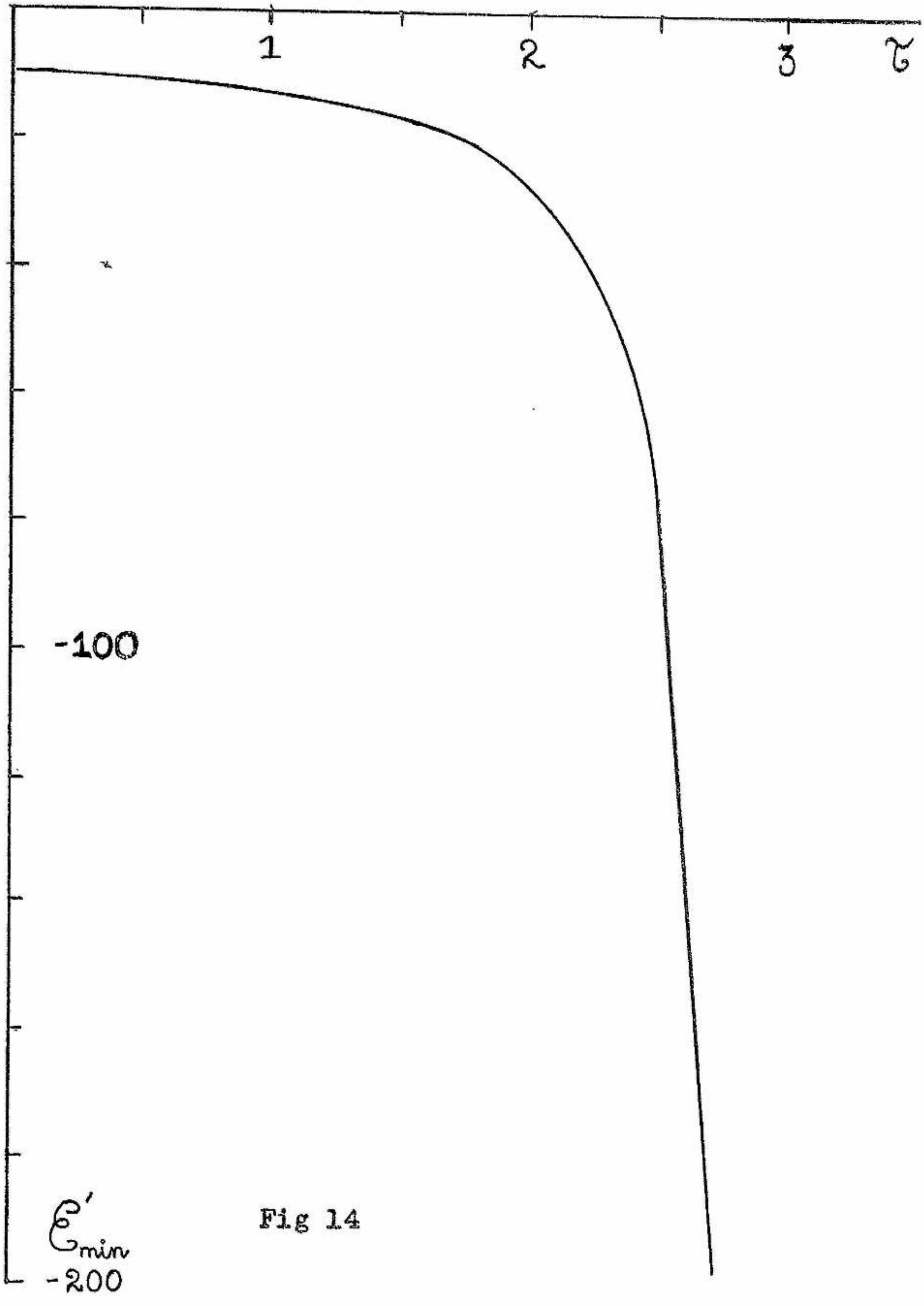
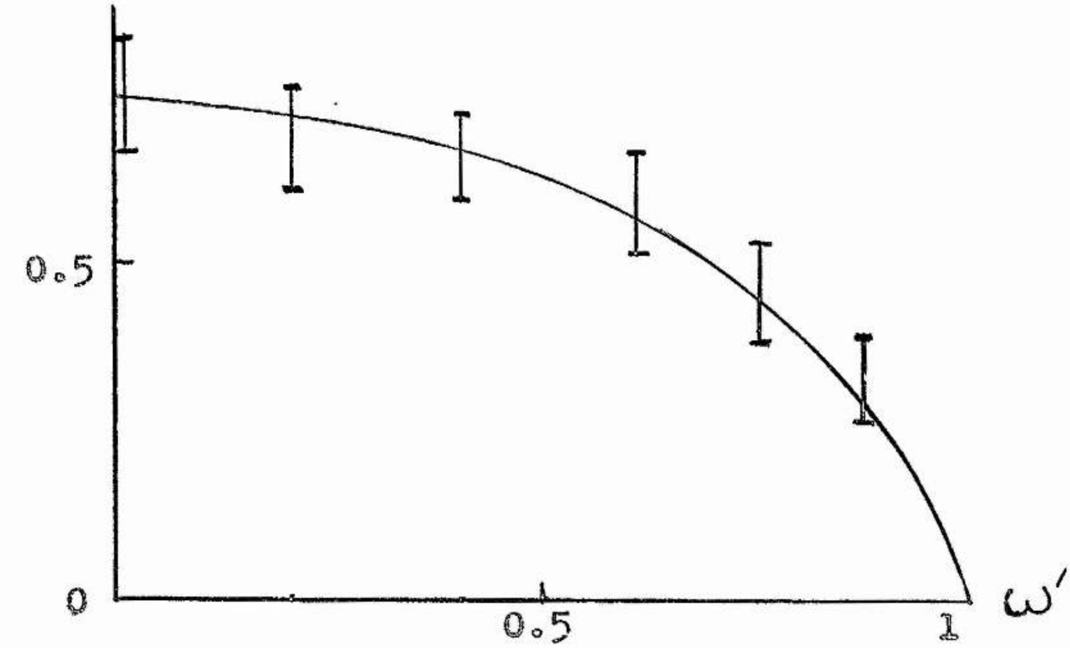


Fig 14

$\mathcal{E}_{min}$ v τ for the singular decay of fig 12

Fig 15b (singular)



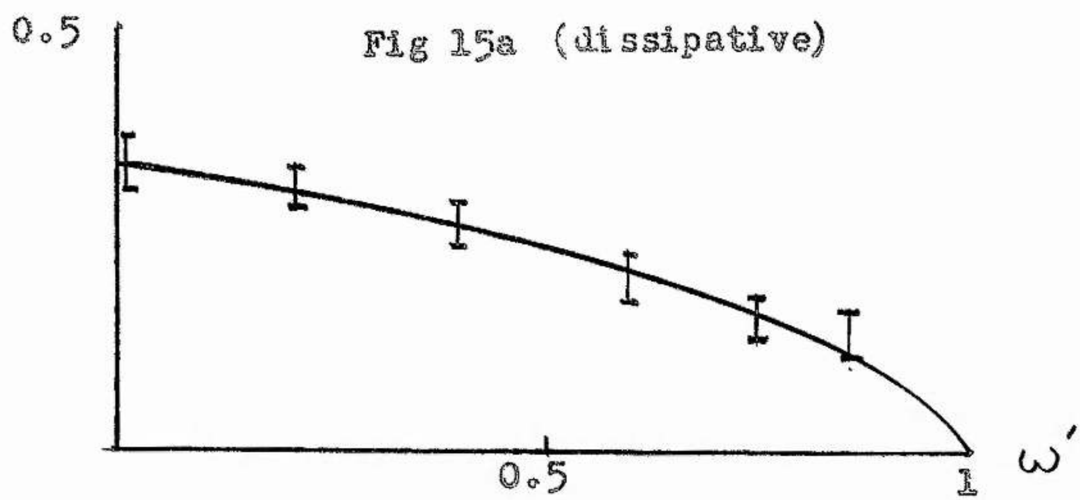


Fig 15a (dissipative)

Plots of the reciprocal mean lifetimes as a function of ω' for the singular and dissipative decay modes. The general shape of these curves is similar to the plot of Ω_i' v ω' given in fig 2, indicating qualititive agreement between the results of chapters 3 and 4.

5. ASSIGNMENT OF PARAMETERS

5.1 Introduction

A particle-like solution in even a simple nonlinear field theory can be considered to possess several physical properties. For example for the field considered a mass and a charge can be ascribed from equations (1.8) and (1.9). The particle can be considered to have some size. For example there is a radius R within which the energy density is finite and non zero, while outside this radius, it is essentially zero. By perturbing the particle it is possible to determine its stability, and if unstable to estimate its lifetime. It is even possible in some crude sense, to determine its decay modes. It is conceivable that by considering multicomponent fields or non-spherically symmetric solutions spin might also be incorporated into the theory.

Despite the fact that the simple field considered is capable of all this, it has serious defects. Let us consider first the case of $\omega' = 0$. From first-order perturbation theory one can estimate the lifetime of the 'particle' to be $1/\Omega_i$ secs. i.e. $\sim (4Kc)^{-1}$ secs. Direct perturbation methods give a fuller account and suggest a decay time of $1.5/Kc$ secs. for the singular decay and of $3/Kc$ secs for a dissipative decay. Let us take an average decay time of $2/Kc$ secs. Then this compares quite favourably with the estimates from first-order perturbation theory (same to within an order of magnitude). The particle size is estimated to be ~ 2 units of r' i.e. $2/K$ units of length, which gives a value to the ratio/

ratio (lifetime/size) of the order of $1/c$, a constant independent of the field parameters K, μ. Further, the value of this ratio $\sim 10^{-10}$ secs/cm., is not in accord with the experimentally obtained values of lifetime/size for the metastable particles (π, K mesons), though it is at least of the correct order of magnitude for the highly unstable mesons (lifetimes $\sim 10^{-23}$ secs) if one assumes that the size of these particles is about one Fermi.

The integral relations (2.1) can be used to reduce the expression for the energy $E = \int \mathcal{E}\, dV$ to

$$E = \frac{2KI}{\mu^2(1-\omega'^2)^{\frac{1}{2}}} \quad \text{where} \quad I = 4\pi\int_0^\infty \varphi'^2 r'^2 dr' \qquad (5.1)$$

Now (1.6) has particle-like solutions with values of the integral I in the ratios 1.5:9.6 : 29.1: . . . and hence this theory predicts the existence of an infinite number of spinless, neutral particles with masses in the above ratios.

Although the lifetime/size ratio is independent of the equation parameters, one can adjust the lifetime of a particle to the experimental value by suitable choice of K. (In the case of π,K mesons fitting K to the lifetimes would however result in unphysically large sizes for these particles). The rest mass of the particle can be matched to experiment by adjustment of μ.

In the time-dependent case one has an extra parameter ω'. It might be thought in this case ($\omega' \neq 0$) that the lifetime/size ratio would/

would be a function of the field parameters but this is not so. The size of the particle is ~ 2 units of r'. But r is proportional to $(1 - \omega'^2)^{-\frac{1}{2}}$, and so the particle becomes more spatially extended as ω' increases. However the lifetime also increases as ω' increases. In chapter 3 it was shown that $\Omega_i'(\omega')$ is to a good approximation given by Barston's limiting curve, and so Ω_i' decreases as $(1 - \omega'^2)^{\frac{1}{2}}$, and hence the lifetime increases as $(1 - \omega'^2)^{-\frac{1}{2}}$. This then implies that the ratio lifetime/size remains to a good approximation, independent of the field parameters K, μ, ω and of the order $1/c$

It had been hoped originally that for certain values of ω' the 'particle' would have been more stable than for other values of ω' and this would then have singled out these particular values of ω'. However, this did not occur and one is left with ω' as an undefined parameter, yielding a continuous spectrum of allowed energies unless ω' can be fixed in some way. Some suggestions as to how to fix ω' are considered.

5.2 Attempts to fix ω

(a) If ω is not zero then the charge Q can be expressed as

$$Q = \int \rho \, dV = \frac{2e\omega' I}{\hbar c \mu^2 (1-\omega'^2)^{\frac{1}{2}}} \tag{5.2}$$

If one now asserts charge quantisation i.e. $Q = \pm e$ then one can determine ω' to be

$$\omega' = \pm \left[1 + \left(\frac{2I}{\hbar c \mu^2}\right)^2 \right]^{-\frac{1}{2}} \tag{5.3}$$

and

$$E = \left[(\hbar c K)^2 + \left(\frac{2KI}{\mu^2}\right)^2 \right]^{\frac{1}{2}} \tag{5.4}$$

For the neutral case ($\omega' = 0$), $E = 2KI/\mu^2$. Thus one gets the satisfactory result that the mass of the charged particle is always greater than that of the neutral particle, but can be made to approximate the mass of the neutral particle if $\mu^2 \ll 2I/\hbar c$.

(b) It would be attractive to identify ω the angular frequency of oscillation of the solution, with the de Broglie frequency ($E/\hbar$). This gives a quadratic equation for ω'

$$\omega' = \frac{2I}{\hbar \mu^2 c (1 - \omega'^2)^{\frac{1}{2}}} \tag{5.5}$$

which has solution

$$\omega'^2 = \frac{1 \pm \sqrt{1 - \left(\frac{4I}{\hbar c \mu^2}\right)^2}}{2} \tag{5.6}$$

Unless $\mu^2 > \frac{4I}{\hbar c}$, (5.6) has no solution for real ω', and unless $\mu^2 = \frac{4I}{\hbar c}$ there will be two solutions for ω'. If one takes the case of $\mu^2 = 4I/\hbar c$ then $\omega' = 1/\sqrt{2}$, $Q = e/2$, $E = K\hbar c/\sqrt{2}$

Comparison of (5.3) and (5.6) shows that there is no value of ω' for which conditions (a) and (b) can be simultaneously satisfied, except $\omega'^2 = 1$ which is not admissible since (1.5) breaks down for this case/

case.

(c) Could there be a special type of solution for $\omega' = 1$?

If one takes a solution of the form (1.3) and substitutes in (1.1) one gets equation (1.4). This has no particle-like solutions for ($K^2 - \omega^2/c^2$) -ve, but perhaps the case $K^2 - \omega^2/c^2 = 0$ i.e. $\omega' = 1$ has. In this case (1.4) becomes

$$\frac{d^2\varphi}{dr^2} + \frac{2}{r}\frac{d\varphi}{dr} = -\mu^2\varphi^3$$

In fact it can be shown that this equation has no particle-like solutions,$^+$ and thus that this approach does not work for this particular field equation.

While none of the suggestions made here are satisfactory for this field, they might give more satisfactory results when applied to other fields. For example there might be a value of ω' for which (a) and (b) could be simultaneously satisfied, or time-dependent particle-like solutions might exist only for special values of ω' as suggested in (c)$^{\neq}$

$^+$ Rosen[11] shows that the equation $\frac{d^2\varphi}{dr^2} + \frac{2}{r}\frac{d\varphi}{dr} = -\mu^2\varphi^n$ has particle-like solutions only when n = 5. In this case $\mu^2 \Rightarrow 3g$ produces the equation Rosen has studied and shown to have a solution

$$\varphi = z/(z^4 g + r^2)^{\frac{1}{2}}$$

$^{\neq}$ It might be thought that the field equation

$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = K^2\psi - 3g(\psi\psi^*)^2\psi$ is an example of this

type/

type. For, putting $\psi = \varphi e^{-i\omega t}$ gives the equation $\nabla^2\varphi = (K^2 - \omega^2/c^2)\varphi - 3g\varphi^5$.

Nehari[10] shows that the equation

$\frac{d^2\varphi}{dr^2} + \frac{2}{r}\frac{d\varphi}{dr} = \varphi - \varphi^5$ can have no particle-like solutions, but if one asserts that $K^2 - \omega^2/c^2 = 0$ then one has the equation $\frac{d^2\varphi}{dr^2} + \frac{2}{r}\frac{d\varphi}{dr} = -3g\varphi^5$ which has a solution $\varphi = z/(z^4 g + r^2)^{\frac{1}{2}}$

However, if ω is non zero, then the energy for this solution is infinite. (When ω is zero, the energy is finite). Thus the case $\omega \neq 0$ does not give rise to particle-like solutions of this form.

5.3 Defects of this field

This field has been found to be unsatisfactory in several respects. When disturbed, the particle-like solutions of the type examined always decay, and do so very rapidly. The ratio of the lifetime/size for such solutions is approximately a constant, independent of the parameters of the theory K, μ, ω. Not only that but the value for this ratio is far removed from that of the metastable particles, though it might be acceptable for resonance-type particles.

The energy density is not positive definite. Since we want to consider a particle as a local concentration of energy, it would be preferable to have $\mathcal{E}$ everywhere positive. A direct consequence of having a field for which $\mathcal{E}$ can be negative, is that singular decays are possible. The particle decays by drawing an increasing amount of negative energy into a localised region around the origin. It is difficult to imagine this mode of decay as being physically sensible although/

although the other method of decay, the dissipative mode, whereby the particle radiates all its energy, seems acceptable.

The theory involves three parameters K, μ, ω. No satisfactory method of fixing these parameters has been found though some tentative suggestions have been made. It was originally hoped that a time-dependent particle-like solution of the form (1.3) might be more interesting than a time-independent solution. But this has not in fact been so for this particular field. A generalisation of this field will now be considered which overcomes some, though not all, of the aforementioned defects.

6. A GENERALISED FIELD

6.1 Introduction

The field equation considered in chapters 2-5 has been shown to be unsatisfactory from several points of view, and we now want to consider the stability of particle-like solutions to another field equation. Some simple equations can be ruled out as not having particle-like solutions. The equation

$$\frac{d^2\theta}{dr^2} + \frac{2}{r}\frac{d\theta}{dr} = \theta - \theta^5$$

has been shown by Nehari[10] to have no particle-like solutions, while Rosen[11] has shown that the equation

$$\frac{d^2\theta}{dr^2} + \frac{2}{r}\frac{d\theta}{dr} = -\theta^n$$

has no satisfactory solutions other than for n = 5.

We want a field equation which might give particle-like solutions for which the energy density is positive definite, and for which the lifetime/size ratio is not independent of the field parameters. A possible candidate is

$$\frac{d^2\theta}{dr^2} + \frac{2}{r}\frac{d\theta}{dr} = \theta - \theta^3 + B\theta^5 \tag{6.1}$$

where B is some real parameter. For B = 0, (6.1) reduces to (1.6) whose solutions are known. An equation such as (6.1) can be obtained in the following way.

One starts with a Lagrangian density

$$\mathcal{L} = \frac{1}{c^2}\left|\frac{\partial\psi}{\partial t}\right|^2 - |\nabla\psi|^2 - K^2|\psi|^2 + \frac{\mu^2}{2}|\psi|^4 - \frac{\lambda}{3}|\psi|^6 \tag{6.2}$$

where K, μ, λ are real parameters and at this stage λ can be positive or negative. Later we will concentrate on the case $\lambda > 0$. The Euler-Lagrange equation for the Lagrangian (6.2) is

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial\psi}{\partial t^2} = K^2\psi - \mu^2\psi\psi^*\psi + \lambda(\psi\psi^*)^2\psi \qquad (6.3)$$

while the energy density can be expressed as

$$\mathcal{E} = \frac{1}{c^2}\left|\frac{\partial\psi}{\partial t}\right|^2 + |\nabla\psi|^2 + K^2|\psi|^2 - \frac{\mu^2}{2}|\psi|^4 + \frac{\lambda}{3}|\psi|^6 \qquad (6.4)$$

As for the previous field equation, we try a solution of the form

$$\psi = \varphi(r)\, e^{-i\omega t} \qquad (1.3)$$

where φ is a real and spherically symmetric function of r. If one follows the procedure for the $\lambda = 0$ case, using the transformations (1.5), then (6.3) can be reduced to

$$\frac{d^2\varphi'}{dr'^2} + \frac{2}{r'}\frac{d\varphi'}{dr'} = \varphi' - \varphi'^3 + B\varphi'^5 \qquad (6.5)$$

where B is defined by $B = \lambda K^2(1-\omega'^2)/\mu^4$

The same transformations (1.5) and (1.3) allow one to express the energy density for a solution of (6.5) of the form (1.3) as

$$\mathcal{E} = \frac{K^4}{\mu^2}(1-\omega'^2)^2\left\{\left(\frac{d\varphi'}{dr'}\right)^2 + \frac{(1+\omega'^2)}{(1-\omega'^2)}\varphi'^2 - \frac{1}{2}\varphi'^4 + \frac{B}{3}\varphi'^6\right\} \qquad (6.6)$$

Solutions to (6.5) will now be considered. In section 6.2 some of the properties/

properties of the $B = 0$ case are reconsidered and analogies with the present case exploited. In sections 6.3 and 6.4 a more rigorous method of analysis is given. Some particle-like solutions are then found numerically in section 6.5.

6.2 Analogies with (1.6)

We recall the pertinent results for the $B = 0$ case. There is a countable infinity of particle-like solutions having respectively 0, 1, 2 . . . nodes. For small r', φ' can be expanded as a power series

$$\varphi' = a_0 + a_2 r'^2 + a_4 r'^4 + \ldots$$

For certain values of a_0, denoted $A(k)$, $k = 0, 1, 2 \ldots$ then φ' tends monotonically to zero as $r' \to \infty$. For a_0 not equal to $A(k)$, φ' oscillates about the special solutions $\varphi' = \pm 1$, depending on whether $a_0 \gtrless A(k)$, and k is odd or even.

Alternatively, solutions with the asymptotic form $de^{-r'}/r'$ will have $(d\varphi'/dr')_{r'=0}$ equal to zero only for certain values of d, denoted $D(k)$; $k = 0, 1, 2, \ldots$ We now look for the counterparts of these results in the $B \neq 0$ case.

Equation (6.5) has the special solutions given by the solutions of

$$\varphi' - \varphi'^3 + B\varphi'^5 = 0$$

i.e. $\varphi' = 0$, $\varphi' = \pm A_1$, $\varphi' = \pm A_2$ where A_1 and A_2 are defined/

defined by

$$A_1 = \sqrt{\frac{1 - \sqrt{1-4B}}{2B}} \qquad A_2 = \sqrt{\frac{1 + \sqrt{1-4B}}{2B}} \quad (6.7)$$

Four separate cases can be defined.

(i) $B > 1/4$ Both A_1 and A_2 are complex. The only real special solution is $\varphi' = 0$

(ii) $0 < B < 1/4$ A_1 and A_2 are both real. Thus there are four nontrivial special solutions $\varphi' = \pm A_1$, $\varphi' = \pm A_2$

(iii) $-\infty < B < 0$ A_1 real, A_2 complex. There are only two real nontrivial special solutions $\varphi' = \pm A_1$.

(iv) $B = 0$ A_1 real $= 1$. This case is essentially the same as (iii).

In analogy with the $B = 0$ case we expect that there will exist certain discrete values $A(k)$ such that if $a_0 \neq A(k)$ then the solution φ' will oscillate about the lower special solutions i.e. about $\pm A_1$. In the $B = 0$ case there was no other finite special solution, corresponding to A_2, as there is when $0 < B < 1/4$. It is shown in Appendix K that if $a_0 > A_2$, then φ' can not oscillate about $\pm A_1$, or tend to zero monotonically. This implies that $A(k)$ is less than A_2 for all k for all B in the range $0 < B < 1/4$. It is also shown in Appendix K that there can be no particle-like solutions for $B > 1/4$. For $B < 0$ we expect the behaviour to be similar to the $B = 0$ case, only as B becomes more negative, A_1 decreases from 1 to zero.

6.3 Phase space analysis of (6.5)

In/

In this section a more rigorous proof of the predictions of section 6.2 is given by analysing (6.5) in phase space. It is shown that particle-like solutions can not exist for $B > 3/16$, but probably do so for all B in the range $-\infty < B < 3/16$.

In order to exploit the analogy with forces on macroscopic point particles and their equations of motion, (6.5) is rewritten for this section as

$$\ddot{y} + \frac{2}{t}\dot{y} = y - y^3 + By^5 \qquad (6.8)$$

where . denotes differentiation with respect to t.

Equation (6.8) can be recast as

$$\frac{d}{dy}\left(\tfrac{1}{2}\dot{y}^2\right) = y - y^3 + By^5 - 2\dot{y}/t \qquad (6.9)$$

Were it not for the term $\left(-\frac{2}{t}\dot{y}\right)$, (6.9) could be integrated immediately.

Let us consider the following equation

$$\frac{d}{dy}\left(\tfrac{1}{2}\dot{y}^2\right) = y - y^3 + By^5 \qquad (6.10)$$

which is derived from (6.9) by omitting the troublesome term.

Integrating (6.10) gives

$$\tfrac{1}{2}\dot{y}^2 + \left(-\tfrac{1}{2}y^2 + \frac{1}{4}y^4 - \frac{B}{6}y^6\right) = C \qquad (6.11)$$

where C is some constant of integration. Equation (6.10) describes the conservative motion of a particle of unit mass under the force $(y - y^3 + By^5)$. The energy of the particle can then be expressed as (6.11) where $C = T + V$. (T is the kinetic energy and V the potential energy)./

energy).

$$V = -\tfrac{1}{2}y^2 + \tfrac{1}{4}y^4 - \tfrac{B}{6}y^6 \qquad (6.12)$$

The equilibrium points of this motion are given by

$$\frac{\partial V}{\partial y} = 0 = -y + y^3 - By^5 \qquad (6.13)$$

i.e. $y = 0$, $y = \pm A_1$, $y = \pm A_2$ which points are just the special solutions of (6.8). It can be shown that $y = \pm A_1$ are stable points and $y = \pm A_2$ are unstable points. Now it is not (6.10) we want to solve but (6.9). Integration of (6.9) gives

$$\tfrac{1}{2}\dot{y}^2 - \tfrac{1}{2}y^2 + \tfrac{1}{4}y^4 - \tfrac{B}{6}y^6 = -\int \frac{2}{t}(\dot{y})^2\,dt$$

which can be expressed as

$$\frac{d\Upsilon}{dt} = -\frac{4}{t}\dot{y}^2 \qquad \Upsilon = 2C \qquad (6.14)$$

where the "energy" C given by (6.11) is no longer constant in time. Since y is real, $(\dot{y})^2$ must be positive, and hence $\frac{\partial \Upsilon}{dt}$ must always be negative. This means that the trajectories of the motion (6.9) must always move to decreasing Υ. We now consider various phase-space trajectories.

6.4 The Phase-space trajectories

Different values of B produce different types of phase trajectories and it is convenient to split the analysis into four separate regions (i) $B = 0$ (ii) $B < 0$ (iii) $0 < B < 3/16$ (iv) $B > 3/16$. We begin by examining/

examining the B = 0 case.

(i) $B = 0$

In fig. 16 we show the appropriate phase space diagram. The solid lines are the curves Υ = constant, and are therefore the trajectories of a particle obeying (6.10). We want the phase trajectory of a particle obeying (6.9), and these are given by (6.14), i.e. the trajectory always moves to decreasing Υ. The figure of eight lying round the origin is the curve $\Upsilon = 0$. Once a trajectory enters such a region it can not escape and must end up on the lowest point in the lobe which it enters i.e. ± 1. For certain values of y (when $\dot{y} = 0$), there are trajectories which go to the origin. These values of y are A(k) k = 0, 1, 2 . . The trajectories for the two lowest order solutions to (6.9) for B = 0 are marked on fig. 16 The trajectories of the higher order solutions can be obtained by a simple extension of the above process.

(ii) $B < 0$

The phase space for B-ve, is very similar to that for the B = 0 case except that $A_1 < 1$. Otherwise the analysis goes through as for the B = 0 case. (The above process would break down if the point P_1 at which the $\Upsilon = 0$ trajectory cuts the line $\dot{y} = 0$ were to lie nearer the origin than A_1. However it is easy to show that this cannot occur for any B).

(iii) $0 < B < 3/16$

When B is positive, A_2 is real and finite and thus one expects the phase space for this case to be more involved. In fig. 17 a typical/

typical phase space for $0 < B < 3/16$ is drawn. From this diagram it is easy to see how the results of section 6.2 fit in. Let us split up the analysis into 3 categories.

(a) $a_o < P_1$

Then the trajectory lies wholly within the hatched region ($\Upsilon < 0$) since the trajectory starts within the $\Upsilon < 0$ region and can therefore not escape. It will spiral around $+A_1$ if a_o is chosen +ve.

(b) $a_o > A_2$

For $a_o > A_2$, the direction of decreasing Υ is away from the origin and such a trajectory can never cross the line $y = A_2$. Thus there are no particle-like solutions for any $A(k) > A_2$. This result is also proved in Appendix K.

(c) $P_1 < a_o < A_2$

All $A(k)$ must lie between P_1 and A_2. Some trajectories for values of a_o in this range are shown in fig.17. The behaviour is essentially the same as for the B -ve and B = 0 cases i.e. except for certain values of a_o, $[A(k)]$, the trajectories enter the hatched regions, from which they cannot escape, and end up spiralling around $\pm A_1$.

(iv) $B > 3/16$

The diagram of fig.17 is typical only of B in the range $0 < B < 3/16$. Let us define P_1 and P_2 as the points at which the $\Upsilon = 0$ trajectory cuts the line $\dot{y} = 0$. This gives

$$P_1 = \sqrt{\frac{\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{4B}{3}}}{\frac{2B}{3}}} \qquad P_2 = \sqrt{\frac{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{4B}{3}}}{\frac{2B}{3}}} \qquad (6.15)$$

In fig 18 we draw A_1, P_1, A_2, P_2 as a function of B. From this graph it is clear that the order of points 0, A_1, P_1, A_2, P_2 is preserved up to $B = {}^3/16$. For $B > {}^3/16$, no real P_1, P_2 exist and a typical phase space for the case ${}^3/16 < B < {}^1/4$ shown in fig. 19. It is clear that for any $a_o < A_2$, we are starting within a region for which Υ is negative, and because the trajectory always moves to decreasing Υ the origin (at which Υ is zero) can never be reached. For $a_o > A_2$ we move in a direction away from the origin to a region of -ve Υ.

Thus there are no values of a_o for which particle-like solutions can occur. i.e. no A(k) exist.

For $B > {}^1/4$, we have already shown that particle-like solutions cannot exist.

Conclusion

From an analysis of (6.10) in the phase plane, it has been shown that particle-like solutions can exist for B in the range $-\infty < B < {}^3/16$, but that such solutions cannot exit for $B > {}^3/16$. When solutions do exist, the values of A(k) can be bounded.

$$P_1 < A(k) \qquad \text{for } -\infty < B < 0$$
$$P_1 < A(k) < A_2 \qquad \text{for } 0 < B < {}^3/16$$

When $a_o \neq A(k)$, the trajectory oscillates about one of the special solutions $\pm A_1$ depending on whether $a_o \gtrless A(k)$.

6.5 Numeric Results

Because/

Because of the similarity between (6.5) and (1.6), it was thought that the same methods as used to solve (1.6) would carry over. This was generally true except that for $B > 0$, the inward integration method broke down. However, in this region the outward, or matching-in-the-middle methods using a minimisation correction routine worked well because A_2 provided a very useful bound to $A(k)$. For example, for $B = 0.1$, $A_2 = 2.978$ while $A(o)$ was found numerically to be 2.9057. As $B \to {}^3/16$, the bound A_2 became ever more severe. For $B < 0$, there is no upper bound A_2, but all 3 methods work for this case. Previously we explained that this field was chosen because it might give solutions with positive definite energy density. It will later be shown that this can only occur if B is greater than zero, and so we concentrate on the nodeless solutions to (6.5) for $B > 0$. In table 4 the values of $A(o)$ and A_2 are given for various values of B.

A further hope for this generalised field is that it will give rise to solutions for which the lifetime/size ratio is a function of at least one of the field parameters. In fig. 20 the nodeless solutions of (6.5) are plotted for some values of B, showing how strongly dependent the size of the solution is on B. The lifetimes of the nodeless solutions as a function of B will be examined in the next chapter.

6.6 Summary

Particle-like solutions to the generalised field are considered. It is shown that such solutions cannot exist for $B > {}^3/16$. The nodeless solutions/

solutions are found for various values of B, particularly +ve B because we are ultimately interested in the case of $\mathcal{E}$ +ve definite which can only occur for B +ve. The parameter B is shown to have a strong effect on the size of the solutions, and it is hoped that particle-like solutions for which the lifetime/size ratio is a function of the field parameters will be obtained.

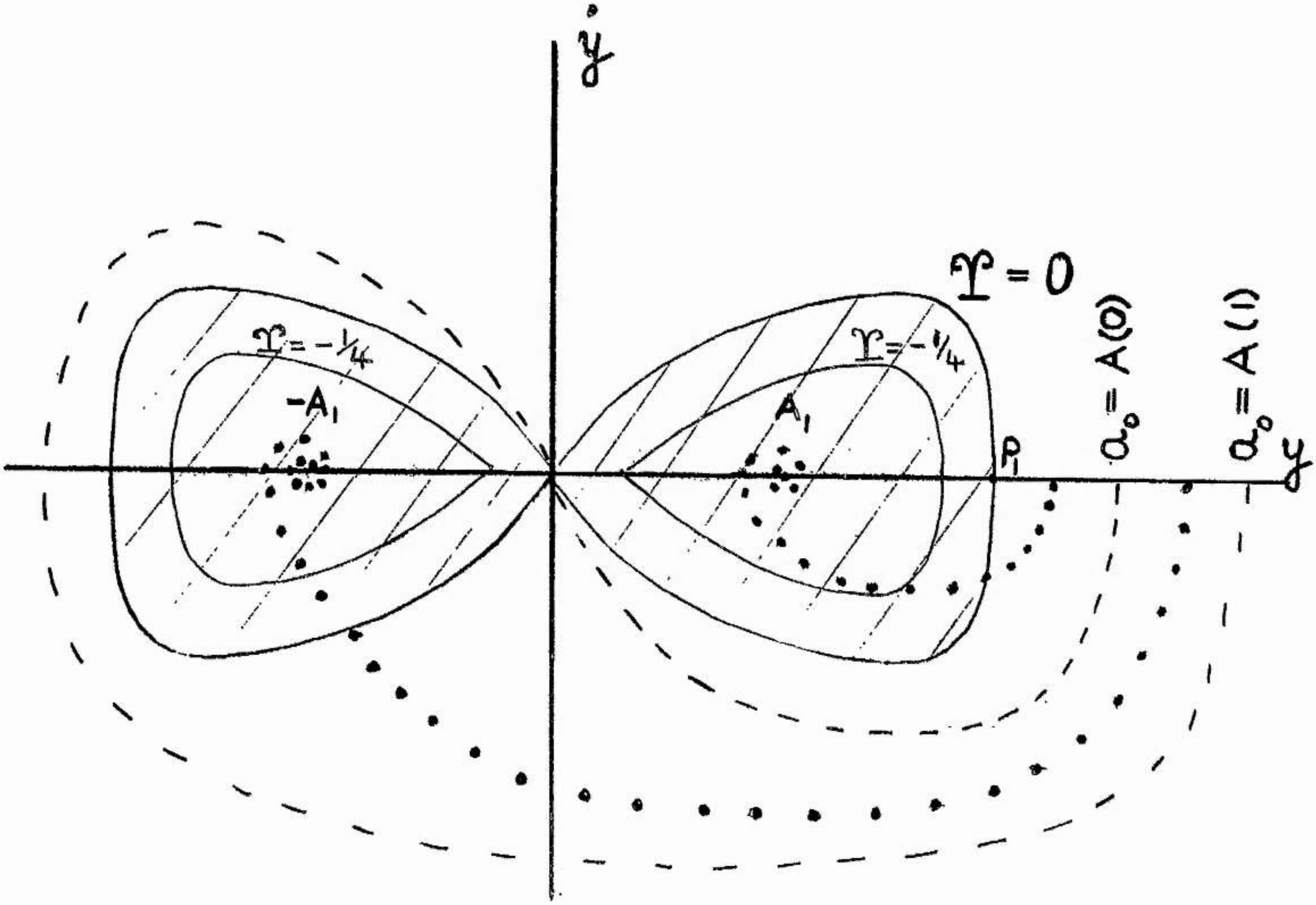


Fig 16

Phase space of (6.9) for the case B= 0, showing the Υ = 0 trajectory(solid line), and the trajectories of the nodeless and one node particle-like solutions (dashed curves). For $a_0 \neq A(k)$, the trajectories (dotted curves) enter the Υ -ve region (hatched area) from which they can not escape, but oscillate asymptotically about the special solutions $\pm A_1$.

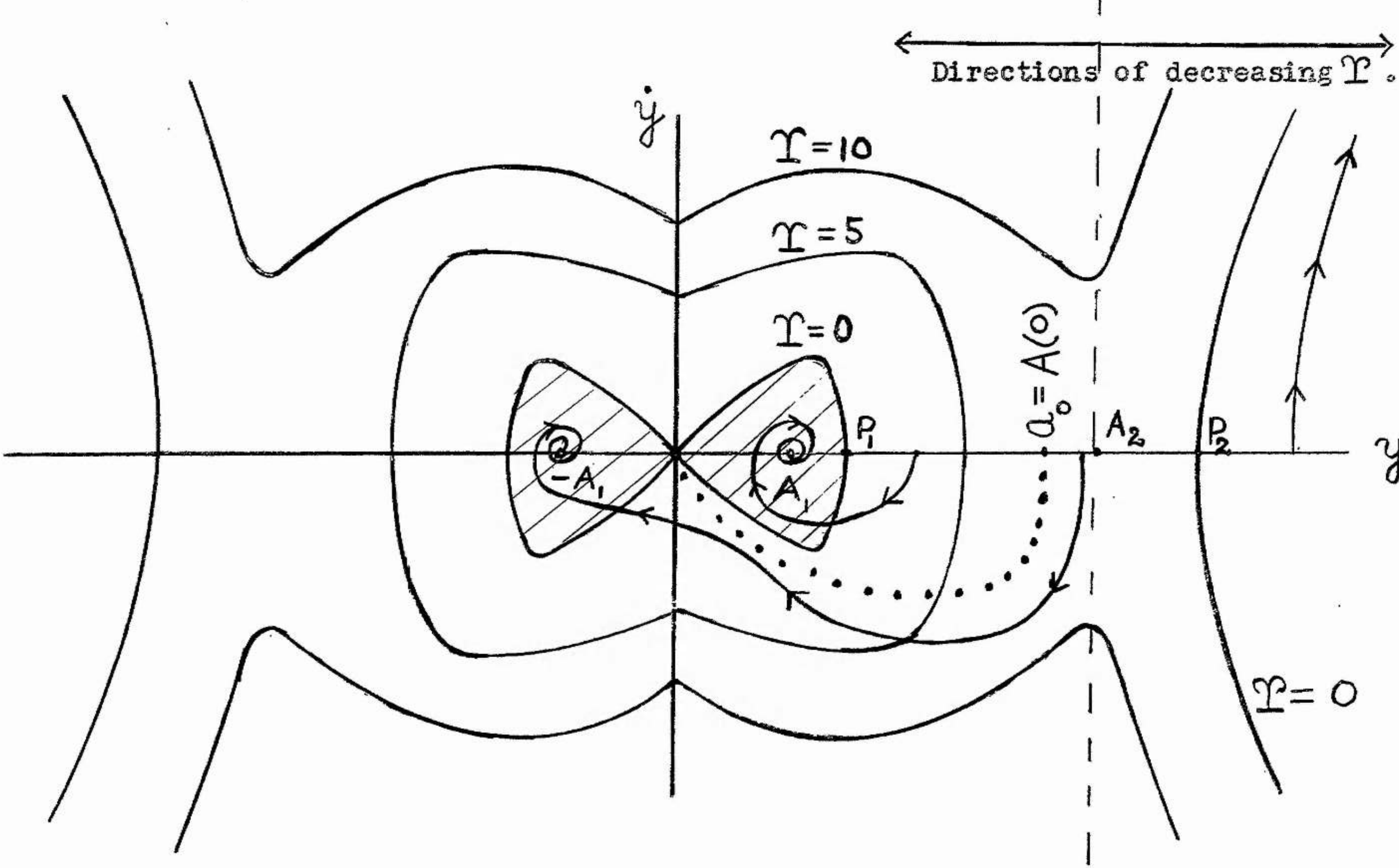


Fig 17

A typical phase space for $0 < B < 3/16$

Dotted curve shows the trajectory of the nodeless particle -like solution. Arrowed curves show other trajectories. Particle-like solutions can only exist if $P_1 < a_0 < A_2$.

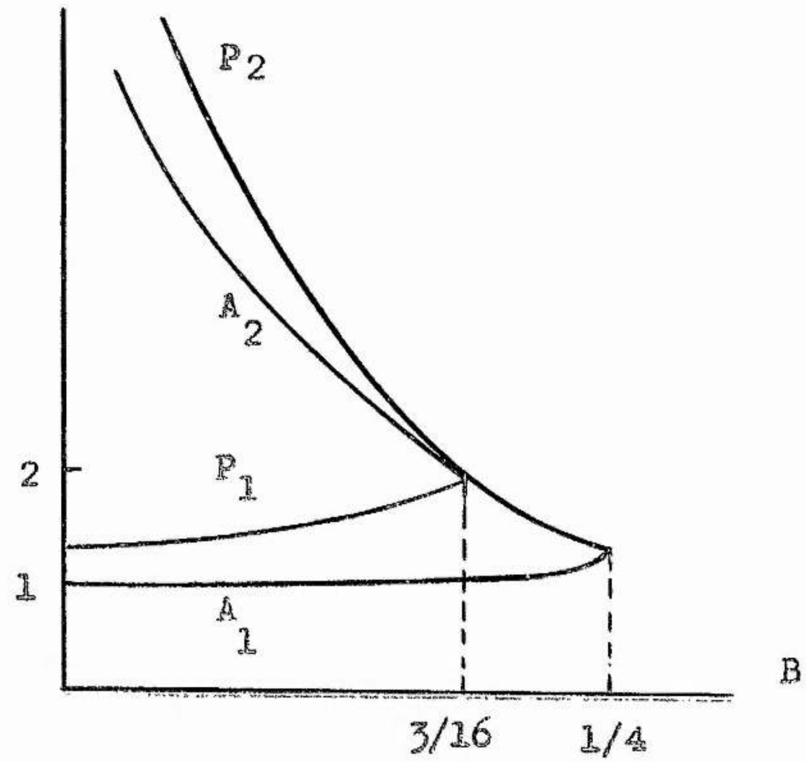


Fig 18

Graph of A_1, P_1, A_2, P_2 v B. The order of points is preserved up to $B = 3/16$, and fig 17 gives a typical phase space for B in the range $0 < B < 3/16$. For $3/16 < B < 1/4$, real P_1 and P_2 do not exist and a typical phase space for this case is shown in fig 19. For $B > \frac{1}{4}$, none of A_1, A_2, P_1, P_2 exist.

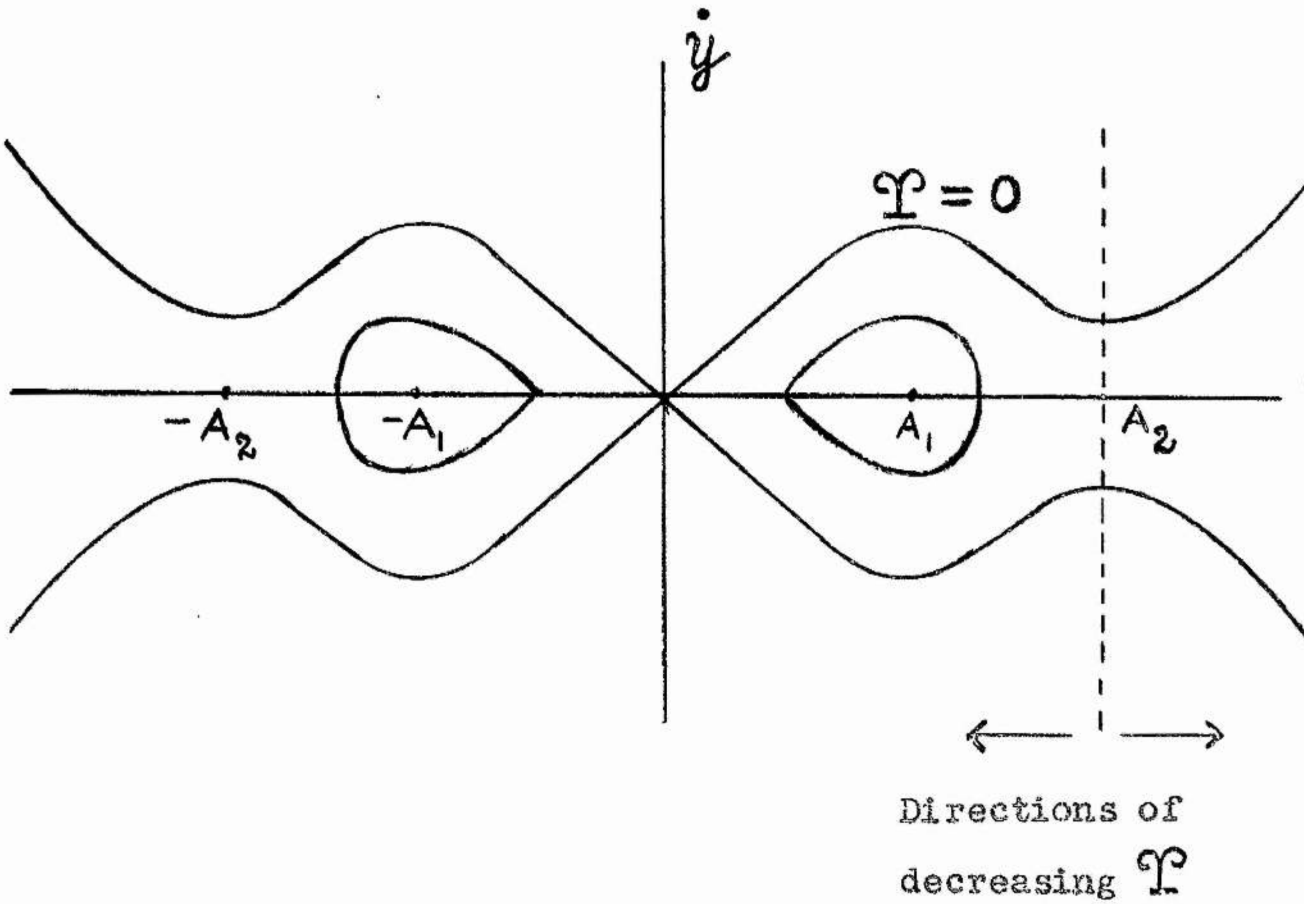


Fig 19

A typical phase space for B in the range $3/16 < B < 1/4$. The line $\dot{y} = 0$ lies wholly (except for the origin) within a region of -ve Υ. Since the trajectories of (6.9) move to regions of decreasing Υ there can be no particle-like solutions to (6.9) since no solution can start with Υ -ve, move continuously to decreasing Υ, and end up at $\Upsilon = 0$ (the origin).

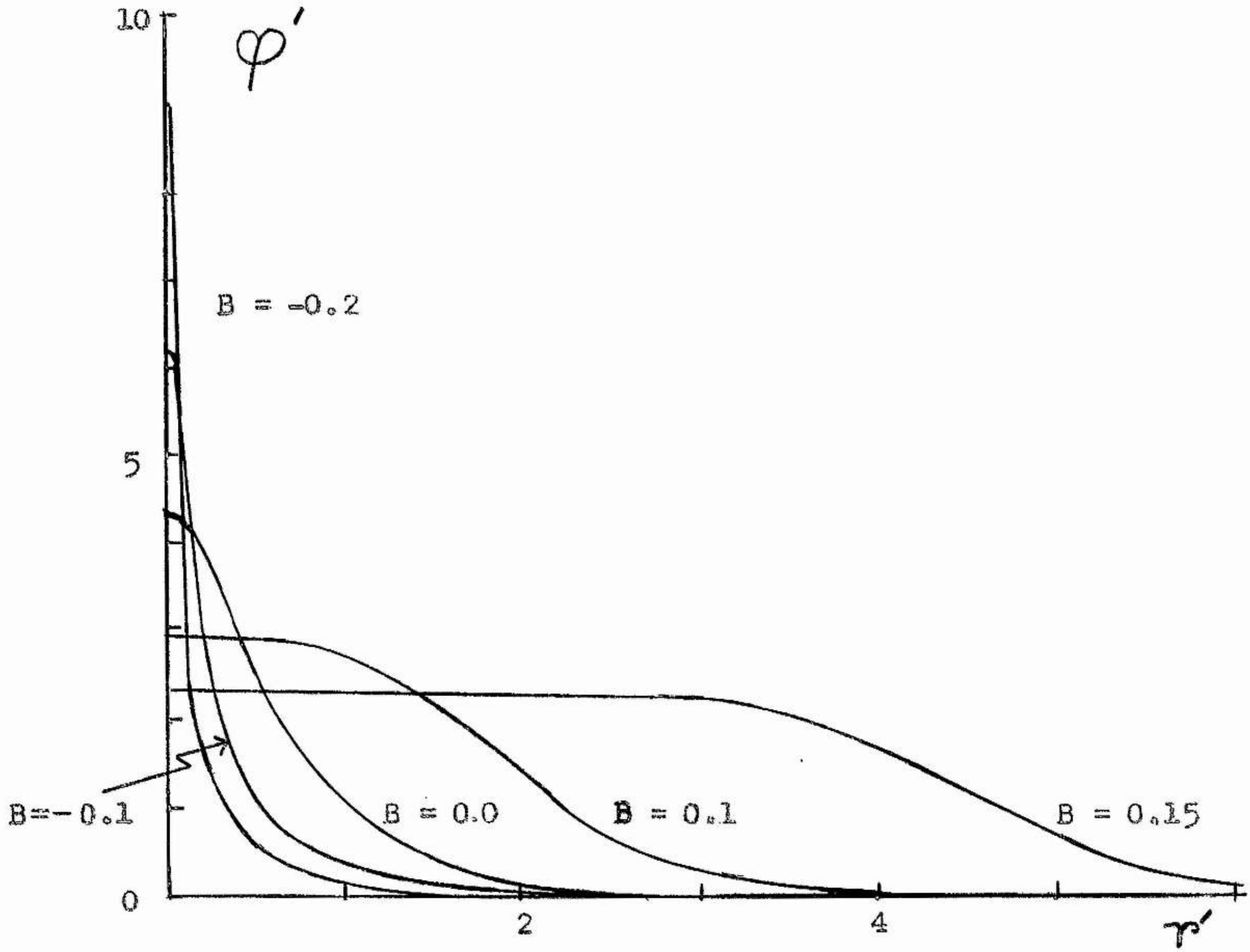


Fig 20

Graph of the nodeless solution of (6.5) for the values of B, -0.2, -0.1, 0.0, 0.1, 0.15 showing how strongly the size and shape of the solution are dependent on B.

Table 4

Table of values of $A(o)$, A_2, $D(o)$ for the nodeless solution of (6.5) for a series of values of B. Note how as $B \rightarrow 3/16$, $A(o) \rightarrow A_2 \rightarrow 2$. For $B > 0.1$, A_2 gives a good upper bound to $A(o)$. For -ve B, a real A_2 does not exist.

B	A(o)	A_2	D(o)
-0.1	6.232 ± .002	-	1.09 ± 0.01
-0.04	5.023 ± .001	-	1.72 ± 0.01
0.0	4.337 ± .001	∞	2.71 ± 0.01
0.05	3.578 ± .001	4.35	5.94 ± 0.04
0.1	2.906 ± .001	2.98	22.7 ± 0.2
0.11	2.781 ± .001	2.82	34.6 ± 0.3
0.12	2.661 ± .001	2.68	58.2 ± 0.5
0.13	2.5456 ± .0005	2.55	113. ± 1.
0.14	2.4358 ± .0003	2.437	279. ± 3.
0.15	2.3325 ± .0001	2.3327	1040. ± 10.
0.16	2.23606 ± .00005	2.236068	8960. ± 20.

7. EXAMINATION OF STABILITY OF PARTICLE-LIKE SOLUTIONS TO THE GENERALISED FIELD

7.1 Introduction

In this chapter, the stability of the nodeless particle-like solution to the field discussed in chapter 6 is examined for $0 < B < 3/16$. The techniques developed in chapter 3 will prove useful in this case. The field equation is given by (6.3). If, as before, we denote by ψ_0 the unperturbed state, by ψ the perturbed state, and by ψ_1 the perturbation and expand (6.3) in terms of ψ_1 keeping only up to first order then we get the equations

$$\nabla^2\psi_1 - \frac{1}{c^2}\frac{\partial^2\psi_1}{\partial t^2} = \left(K^2 - 2\mu^2\psi_0\psi_0^* + 3\lambda\psi_0^2\psi_0^{*2}\right)\psi_1 + \left(-\mu^2\psi_0^2 + 2\lambda\psi_0^3\psi_0^*\right)\psi_1^* \quad (7.1)$$

$$\nabla^2\psi_1^* - \frac{1}{c^2}\frac{\partial^2\psi_1^*}{\partial t^2} = \left(K^2 - 2\mu^2\psi_0\psi_0^* + 3\lambda\psi_0^2\psi_0^{*2}\right)\psi_1^* + \left(-\mu^2\psi_0^{*2} + 2\lambda\psi_0^{*3}\psi_0\right)\psi_1$$

which are the generalisations of (3.1)

If one seeks a solution of (7.1) in the form (3.2) and uses the transformations (1.5), then (7.1) can be reduced to the pair of coupled equations

$$\left(\nabla'^2 + \frac{(\Omega'+\omega')^2-1}{(1-\omega'^2)} + 2\varphi_0'^2 - 3B\varphi_0'^4\right)\eta = \left(-\varphi_0'^2 + 2B\varphi_0'^4\right)\chi \quad (7.2)$$

$$\left(\nabla'^2 + \frac{(\Omega'-\omega')^2-1}{(1-\omega'^2)} + 2\varphi_0'^2 - 3B\varphi_0'^4\right)\chi = \left(-\varphi_0'^2 + 2B\varphi_0'^4\right)\eta$$

where Ω', ω', B are as previously defined, and φ_0' is, for this analysis the nodeless solution of (6.5). Solutions of (7.2) will be examined in much/

much the same way as for the B = 0 case.

When η and χ are spherically symmetric, (7.2) can be expressed as

$$\left(\frac{d^2}{dr'^2} + \frac{(\Omega'+\omega')^2-1}{(1-\omega'^2)} + 2\varphi_0'^2 - 3B\varphi_0'^4\right)g = (-\varphi_0'^2 + 2B\varphi_0'^4)f$$

$$\left(\frac{d^2}{dr'^2} + \frac{(\Omega'-\omega')^2-1}{(1-\omega'^2)} + 2\varphi_0'^2 - 3B\varphi_0'^4\right)f = (-\varphi_0'^2 + 2B\varphi_0'^4)g \qquad (7.3)$$

where $g = r'\eta$ and $f = r'\chi$

Solutions of (7.3) will now be sought.

7.2 Spherically Symmetric Solutions of (7.2)

In chapter 3 it was shown that (3.5) could be written in a form examined by Barston and that consequently Ω_r', Ω_i' could be bounded for the case $\Omega_i' \neq 0$. When Ω_i' was zero, Ω_r' could still be bounded but without recourse to Barston's results. In the generalisation of the field of chapter 3 considered here, (7.3) can still be expressed in the form of (3.12) though with a different definition of H. H is now defined as

$$\begin{bmatrix} -\frac{d^2}{dr'^2} + 1 - 3\varphi_0'^2 + 5B\varphi_0'^4 & 0 \\ 0 & -\frac{d^2}{dr'^2} + 1 - \varphi_0'^2 + B\varphi_0'^4 \end{bmatrix} \qquad (7.4)$$

If Ω_i' is non zero, then Ω_r', Ω_i' can be bounded as

$$\begin{cases} \Omega_r' < \omega' \\ \Omega_r' < \Omega_i'(0)\,(1-\omega'^2)^{\frac{1}{2}} \end{cases} \qquad (7.5)$$

$$\Omega_i' < \Omega_i'(0)\,(1-\omega'^2)^{\frac{1}{2}}$$

where $\Omega_i'(0)$, the maximum value of Ω_i' at $\omega' = 0$, is related to the lowest $(-\Lambda)$ eigenvalue of the operator H by $\Lambda = \Omega_i'^2(0)$. Note that $\Omega_i'(0)$ is a function of B, which, it will be shown, decreases as B increases (algebraically). [Ω_r', Ω_i', ω' are chosen positive in this analysis].

An upper bound to $\Omega_i'(0)$, valid when B is positive is given in Appendix L. To evaluate $\Omega_i'(0)$, one choses a value of B in the range $-\infty < B < 3/16$ and determines the particle-like solution φ_0' for this value of B from (6.5). One then finds the lowest eigenvalue of (7.4) and hence $\Omega_i'(0)$.

When Ω_i' is zero, (7.3) is an essentially real problem and as for the B = 0 case Ω_r' can be bounded by the expression

$$\Omega_r' < 1 - \omega' \qquad \Omega_r',\ \omega' > 0 \tag{7.6}$$

The analysis of (7.3) can conveniently be split up into 4 sections which will be examined separately.

(i) $\omega' = 0$

(ii) $\Omega_r' = 0$, Ω_i' not necessarily zero

(iii) $\Omega_i' = 0$, Ω_r' not necessarily zero

(iv) Neither Ω_r' nor Ω_i' zero.

We are ultimately interested in a particle-like solution for which the energy density is positive definite. Since this can only occur for B +ve, we concentrate on examining (7.3) for B in the range $0 < B < 3/16$.

<u>Examination of (i)</u>

In/

In this case $\omega' = 0$ and (7.3) can be reduced to

$$\left(\frac{d^2}{dr'^2} + (\Omega'^2 - 1) + 3\varphi_0'^2 - 5B\varphi_0'^4\right)z = 0 \qquad (7.7)$$

$$\left(\frac{d^2}{dr'^2} + (\Omega'^2 - 1) + \varphi_0'^2 - B\varphi_0'^4\right)y = 0 \qquad (7.8)$$

where $z = (g + f)$ and $y = (g - f)$.

Equation (7.8) is found to have only the trivial solution $\Omega' = 0$, $y = r'\varphi_0'$. It is found that (7.7) has only one nontrivial solution which is of the form $\Omega_r' = 0$, Ω_i' = non zero. The eigenvalues Ω_i' of (7.7) are given in Table 5 for a series of values of B, showing how Ω_i' decreases, tending to zero, as B tends to $B = 3/16$. In Appendix L, a bound to Ω_i' is deduced, while in fig 21, Ω_i' is plotted against B.

Examination of (ii)

In chapter 3 solutions to (3.5) were found only for $\Omega_r' = 0$. No solutions were found for which Ω_r' was non zero. In analogy with this result, it is expected that solutions of this form will also exist to (7.3). If we a priori seek solutions to (7.3) of the form $\Omega_r' = 0$, then the method of Appendix H is a convenient way of doing so. This method was applied to (7.3) and solutions were indeed found. However, the behaviour of the eigenvalue Ω_i' as a function of ω' for a given B in the range $0 < B < 3/16$ was entirely different from that found for the $B = 0$ case. For a given B, eigenvalues Ω_i' were only found up to some critical value of ω', denoted ω_c'. For ω' in the range $\omega_c' < \omega' < 1$ no solutions of this form were found.

In/

In fig 22 we plot Ω_i' v ω' for values B = .05, .075, .1, .15375. Note that as B increases, so ω_c' decreases. In Table 6 the values of ω_c' for various values of B are given.

Examination of (iii)

Solutions for which Ω_i' is zero, but for which Ω_r' is non zero are now sought. When $\Omega_i' = 0$, equation (7.3) can be reduced to the real problem

$$\left(\frac{d^2}{dr'^2} - \gamma^2 + 2\varphi_0'^2 - 3B\varphi_0'^4\right)g = \left(-\varphi_0'^2 + 2B\varphi_0'^4\right)f \qquad (7.10)$$

$$\left(\frac{d^2}{dr'^2} - \delta^2 + 2\varphi_0'^2 - 3B\varphi_0'^4\right)f = \left(-\varphi_0'^2 + 2B\varphi_0'^4\right)g$$

where γ^2, δ^2 are, as before real positive quantities defined by

$$\gamma^2 = \frac{1-(\Omega'+\omega')^2}{(1-\omega'^2)} \qquad \delta^2 = \frac{1-(\Omega'-\omega')^2}{(1-\omega'^2)}$$

Equation (7.10) has the two solutions

(a) $\gamma^2 = \delta^2 = 1$ $\quad g = -f = r'\varphi_0'$ $\quad$ corresponding to a solution of (7.8)

(b) $\gamma^2 = \delta^2$ $\quad g = f$ $\quad$ corresponding to a solution of (7.7)

In this section we are interested in real solutions for which γ^2 is not equal to δ^2 i.e. for which Ω_r' and ω' are non zero. Equation (3.7) was found to have no such solutions, but in the present case for B > 0, it is found that solutions of this kind do exist for certain ranges of ω'. For a given B, there exists a certain critical value of ω' denoted $\bar{\omega}_c'$ below which no solutions of this form are found. But for $\omega' > \bar{\omega}_c'$ solutions to (7.10) can be found. For each value of ω' there exists a unique value of Ω_r'. The values of Ω_r' v ω' are shown graphically in fig 23 for/

for the values of B, .05, .075, .1, .15375. Further it is found that ω'_c and $\bar{\omega}'_c$ are the same to within numerical error. In table 6 we give the values of ω'_c and $\bar{\omega}'_c$ obtained by methods (ii) and (iii) respectively for the appropriate values of B, and one can see the exceedingly close agreement between these values. This means that a graph of Ω' v ω' for a given B has the form shown in fig 24. (Fig 24 is for the particular case B = 0.1)

Examination of (iv)

When one seeks solution to (7.3) for which neither Ω'_r nor Ω'_i need be zero, (7.3) cannot be simplified and a direct attack on solutions must be made using the techniques described in chapter 3. No solution for which Ω'_r and Ω'_i were simultaneously non zero was found.

Conclusion

For a given B in the range $0 < B < 3/16$, there exists a critical value of ω' denoted ω'_c, such that for $0 < \omega' < \omega'_c$ solutions to (7.3) exist in the form $\Omega'_r = 0$, $\Omega'_i \neq 0$ while for $\omega'_c < \omega' < 1$ the solutions have the form $\Omega'_i = 0$, $\Omega'_r \neq 0$. For any value of ω' the eigenvalue Ω' (either Ω'_r or Ω'_i) is unique and there are no solutions for any ω' for which Ω'_r and Ω'_i are simultaneously non zero,

7.3 Non Spherically Symmetric Solutions

Expansion of η, χ in terms of the spherical harmonics Y_ℓ^m produces the eigenvalue problem

$$\left(\frac{d^2}{dr'^2} + \frac{(\Omega'+\omega')^2 - 1}{(1-\omega'^2)} - \frac{\ell(\ell+1)}{r'^2} + 2\varphi_0'^2 - 3B\varphi_0'^4\right) g_\ell = \left(-\varphi_0'^2 + 2B\varphi_0'^4\right) f_\ell \tag{7.11}$$

$$\left(\frac{d^2}{dr'^2} + \frac{(\Omega'-\omega')^2 - 1}{(1-\omega'^2)} - \frac{\ell(\ell+1)}{r'^2} + 2\varphi_0'^2 - 3B\varphi_0'^4\right) f_\ell = \left(-\varphi_0'^2 + 2B\varphi_0'^4\right) g_\ell$$

This can be recast in the much used Barston form, and so, provided $\Omega_i' \neq 0$, we can again bound Ω_r', Ω_i' by

$$\frac{\Omega_r'^2 + \Omega_i'^2}{(1-\omega'^2)} < -(\text{lowest eigenvalue of H})$$

provided H has a negative eigenvalue.

For (7.11), H is given by

$$\begin{bmatrix} -\frac{d^2}{dr'^2} + \frac{\ell(\ell+1)}{r'^2} + 1 - 3\varphi_0'^2 + 5B\varphi_0'^4 & 0 \\ 0 & -\frac{d^2}{dr'^2} + \frac{\ell(\ell+1)}{r'^2} + 1 - \varphi_0'^2 + B\varphi_0'^4 \end{bmatrix} \quad (7.12)$$

Again, as for chapter 3, it can be shown that H can not have a negative eigenvalue for all ℓ greater than some minimum value. This minimum value depends on B. For example let us take B = 0.1. The lowest eigenvalue of H is then the lowest eigenvalue of

$$-\frac{d^2}{dr'^2} + \frac{\ell(\ell+1)}{r'^2} + 1 - 3\varphi_0'^2 + 5B\varphi_0'^4 \quad (7.13)$$

The eigenfunction must have a point of inflexion at which it is non zero and for this we must have

$$\frac{-\ell(\ell+1)}{r'^2} + 3\varphi_0'^2 - 5B\varphi_0'^4 > 0 \quad (7.14)$$

For B = 0.1, the function $r'^2(3\varphi_0'^2 - 5B\varphi_0'^4)$ is found to have a maximum value of 15.1 and hence (7.14) can be satisfied only for $\ell = 1, 2, 3$. Thus, rigorously, we can state that (7.11) can have no unstable solutions for $\ell > 3$. In fact (7.13) is not found to have any nontrivial eigenvalues giving square integrable eigenfunctions for $\ell = 1, 2$, or 3, and we thus conclude that for B = 0.1, there are no unstable solutions to (7.11).

When/

When we try other values of B we arrive at the same conclusion.

7.4 Discussion

Analysis of the stability of the nodeless particle-like solution of (6.5) reveals some interesting results.

Ω'^2 is found to be always real but may be positive or negative. For a given B $>$ 0, there exists a value of ω' denoted ω'_c at which the sign of Ω'^2 changes.

$0 < \omega' < \omega'_c$	Ω'^2 is negative, giving $\Omega'_r = 0$ and $\Omega' = i\,\Omega'_i$. The fact that Ω'_i is non zero implies that the solution is unstable
$\omega'_c < \omega' < 1$	Ω'^2 is positive, giving $\Omega'_i = 0$ and Ω' = purely real. The fact that Ω'_i is zero implies that the solution is stable.

The suggestion that stable particle-like solutions can exist in this field is examined in greater detail in the next chapter when the stability of particle-like solutions is examined by direct perturbation techniques.

If one considers the $\omega' = 0$ case, one sees from fig 21 that as B tends to its upper limit of $^3/16$, the value of Ω'_i decreases. This implies that the particle takes longer to decay as B increases, but to counteract this the size of the particle also increases (fig 20) and in fact on evaluating the ratio lifetime/size, one finds that it is not strongly/

strongly dependent on B as had initially been hoped. However, we shall see later that when ω' is not zero, this ratio does in fact depend strongly on the field parameters.

So far the results have been presented in terms of the parameter B. This approach is sensible for obtaining results from first-order perturbation theory, since one obtains solutions to the eigenvalue problem (7.3) by choosing a value of B in the legitimate range, deriving the particle-like solution from (6.5) for this value of B and then finding the eigenvalues (Ω_r', Ω_i') of (7.3) for chosen values of ω' for this value of B. Now although B is a useful parameter in obtaining the results we will see that it is more sensible to analyse the results in terms of a new parameter $\tilde{B}$ which we define by

$$\tilde{B} = \frac{\lambda K^2}{\mu^4} = \frac{B}{(1-\omega'^2)}$$

$\tilde{B}$ is thus a parameter independent of ω' which we will see in the next chapter is of significance in direct perturbation methods. When $\omega' = 0$ we have $\tilde{B} = B$.

Having derived the data in terms of constant B we want to convert our results to those of constant $\tilde{B}$. It is convenient to split the analysis into two sections.

(i) $\tilde{B} < 3/16$

(ii) $\tilde{B} > 3/16$

We consider (i). The typical curves of $\Omega_i' \vee \omega'$, $\Omega_r' \vee \omega'$ for constant $\tilde{B}$ are shown in figs 25a and 25b for some values of $\tilde{B} < 3/16$. Note that there again exists a critical value of ω' which we will denote $\tilde{\omega}_c'$.

For/

For $\omega' < \tilde{\omega}'_c$, Ω' is purely imaginary while for $\omega' > \tilde{\omega}'_c$, Ω' is purely real.

The value of $\tilde{\omega}'_c$ is of course intimately related to the value of ω'_c and is a direct consequence of the existence of ω'_c. For example

For B = 0.1, the value of ω'_c is 0.39 ± 0.1

This means that for $\tilde{B} = \dfrac{0.1}{\left[1 - (0.39)^2\right]} = 0.12$, the critical value of ω', i.e. $\tilde{\omega}'_c$ is 0.39.

Let us now consider region (ii). When $\tilde{B} > 3/16$, there is no critical value of ω' in the sense of (i). There is however a lower bound to the range of values of ω' in which particle-like solutions exist. Such solutions can exist only if B $< 3/16$, which implies

$$\left(1 - \frac{3}{16\tilde{B}}\right)^{\frac{1}{2}} < \omega' < 1$$

For all ω' in this range, there exists a value of Ω'_r, but no value for Ω'_i. Curves of Ω'_r v ω' for some values of $\tilde{B}$ are shown in fig 26.

7.5 Summary

In this chapter the stability of the nodeless particle-like solution of (6.5) with time dependence of the form (1.3) was examined for various values of B. It was found that an eigenvalue problem similar to (3.3) occurs but that the nature of the eigenvalue is different. A more convenient method of displaying the results is in terms of the parameter $\tilde{B}$ when two different types of behaviour occur. For a given $\tilde{B} < 3/16$, there exists a critical value of $\tilde{\omega}'_c$, such that for $\omega' < \tilde{\omega}'_c$ the only eigenvalues are of the form $\Omega'_r = 0$; $\Omega'_i \neq 0$, while for $\omega' > \tilde{\omega}'_c$ we find that Ω' is purely real/

real. When $\tilde{B} > {}^{3}/16$, no eigenvalues which are complex exist, the eigenvalue being always purely real. These results suggest that for correctly chosen values of the field parameters, stable time-dependent particle-like solutions exist.

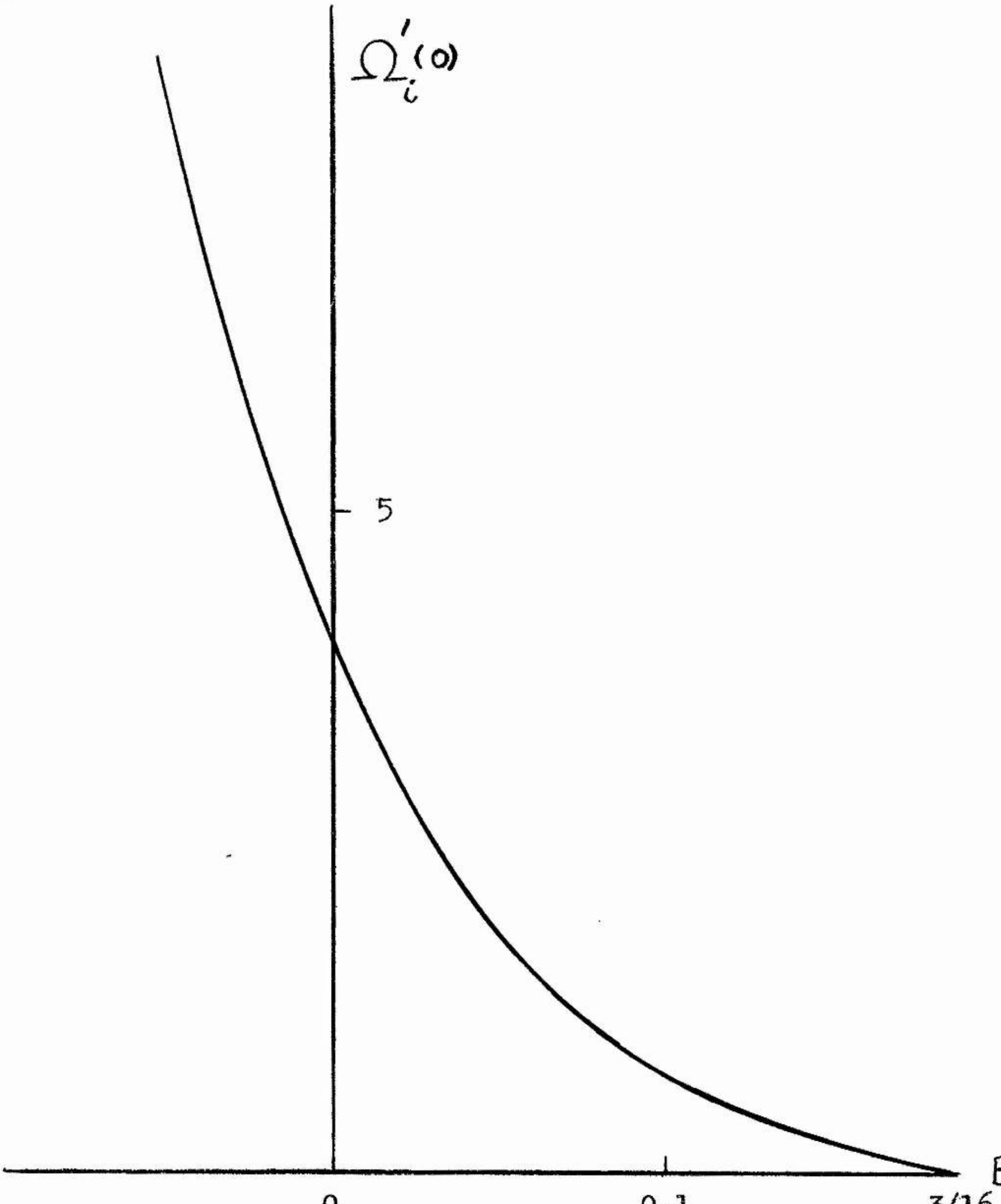


Fig 21

Plot of Ω_i' v B for the case $\omega' = 0$.
Note how Ω_i' decreases as B increases.

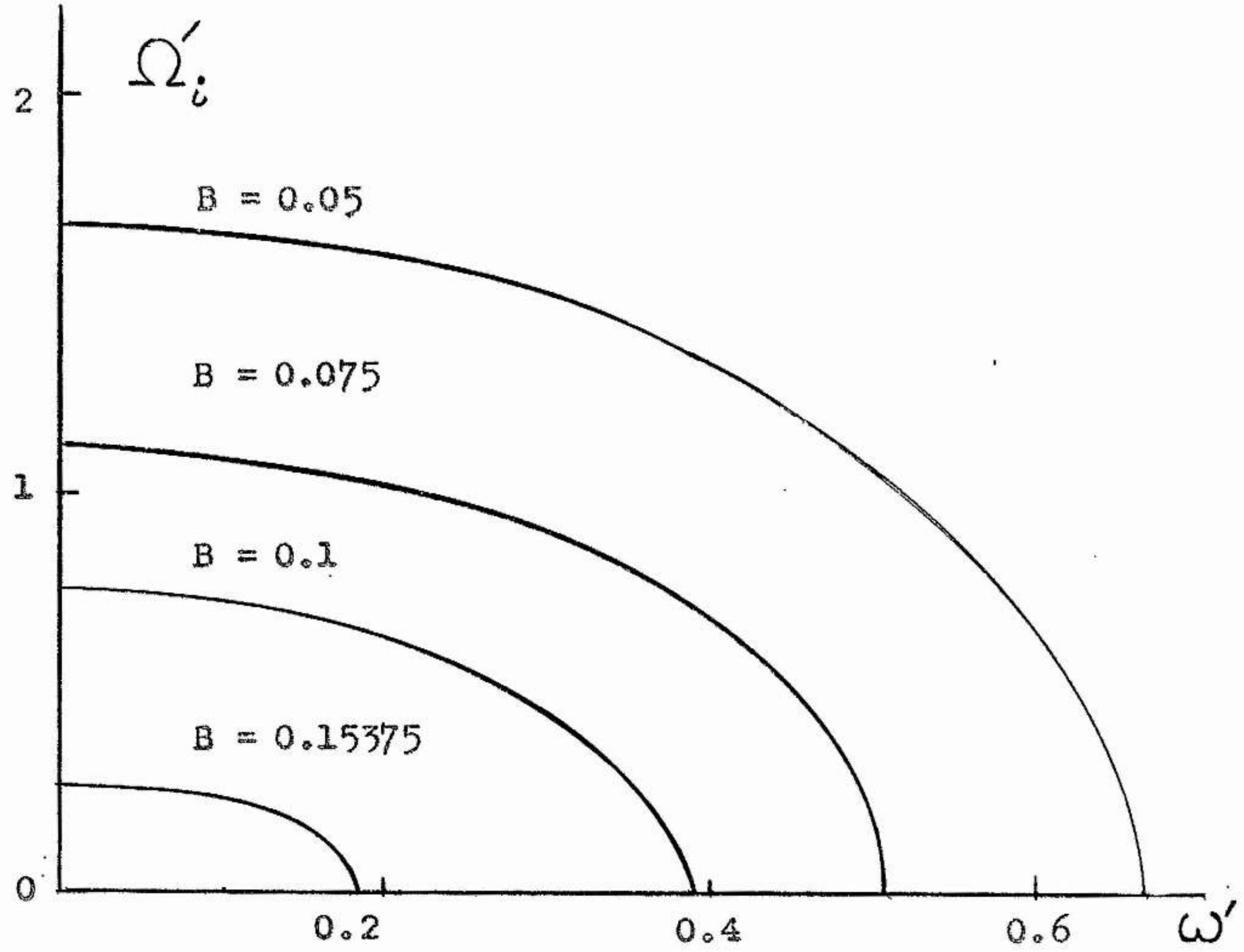


Fig 22

Plot of Ω_i' v ω' for some positive values of B. For each B, there exists a critical value of ω' such that for ω' greater than this value there is no eigenvalue Ω_i'.

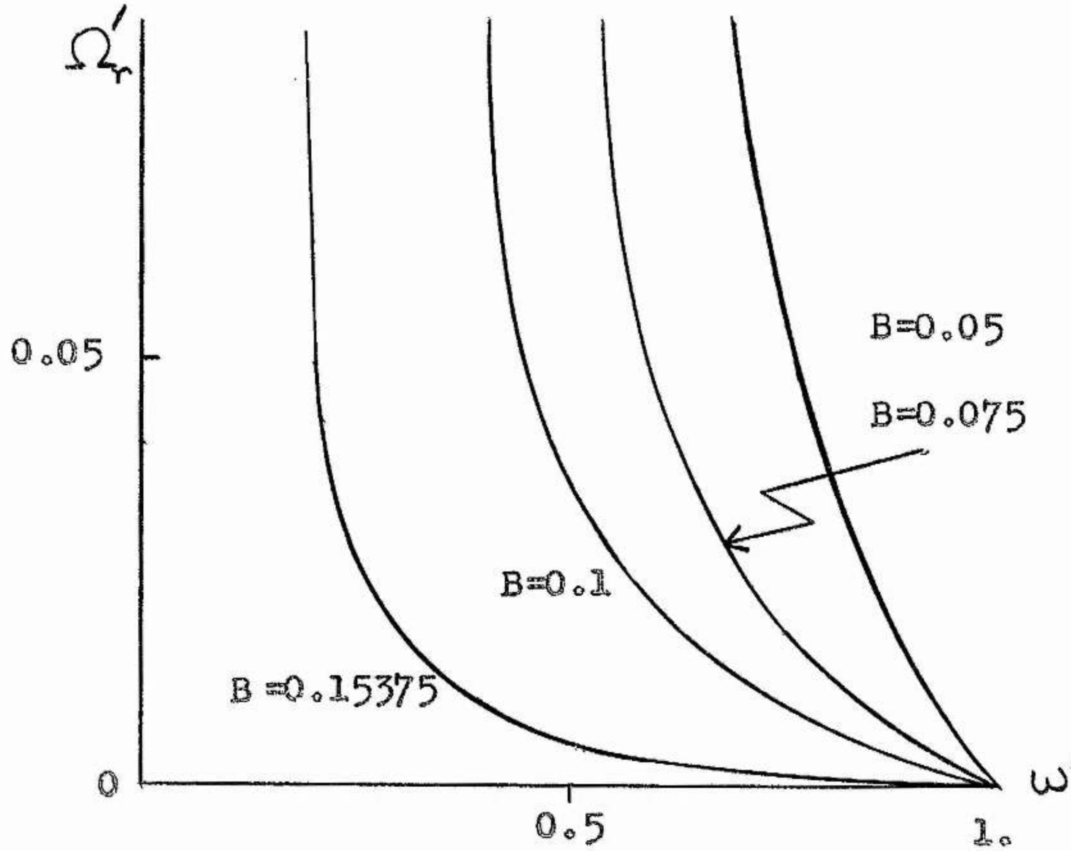
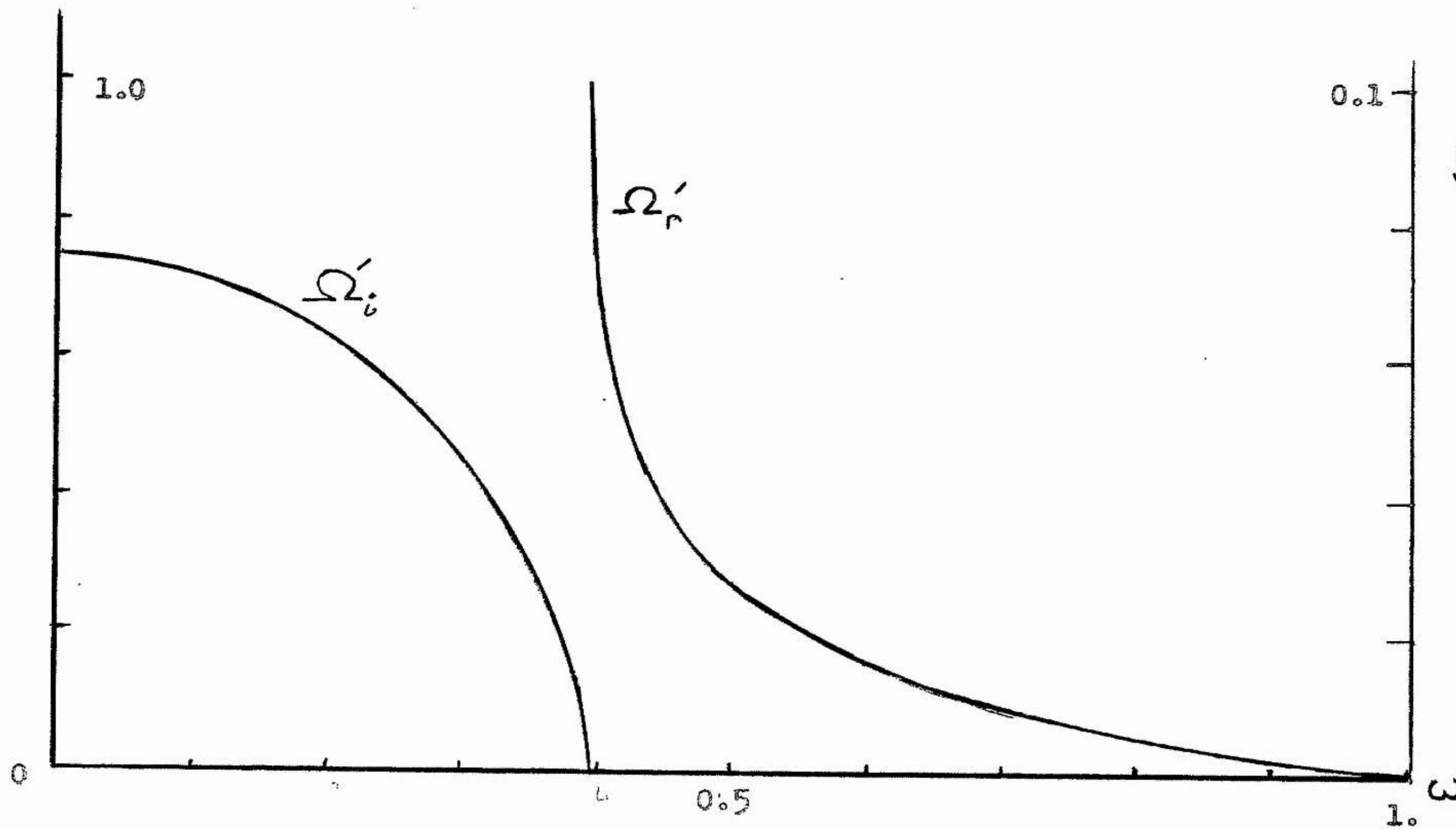


Fig 23

Graph of Ω_r' v ω' for some fixed values of B. For a given B, there exists a critical value of ω', $\bar{\omega}_c'$ below which no solutions of this type exist.

Fig 24

Graph of Ω' v ω' for the case B = 0.1, showing how the nature of the eigenvalue changes at $\omega'_c = 0.39$



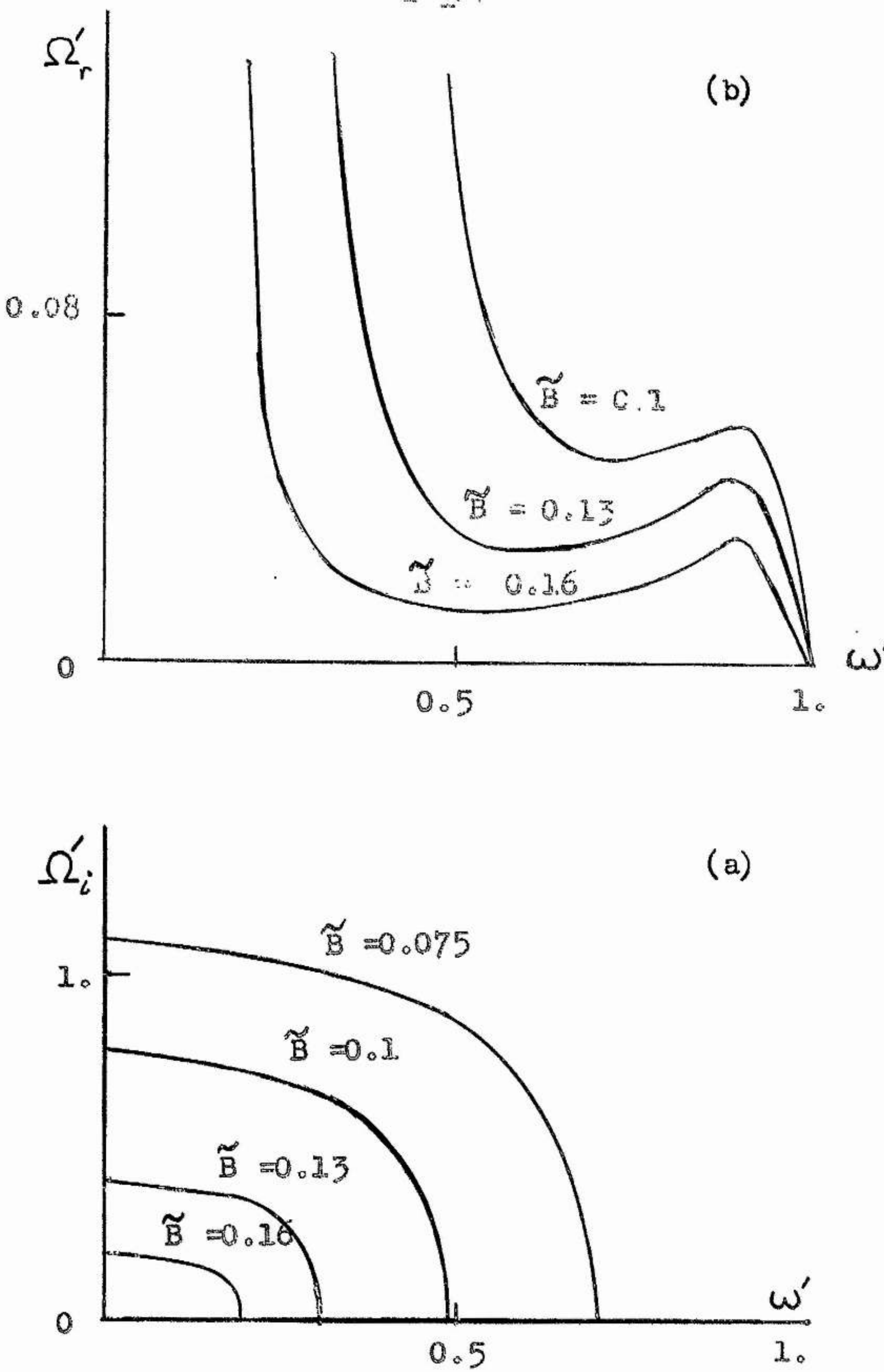


Fig 25

Graphs of Ω'_r v ω' and Ω'_i v ω' for values of $\tilde{B} < 3/16$.

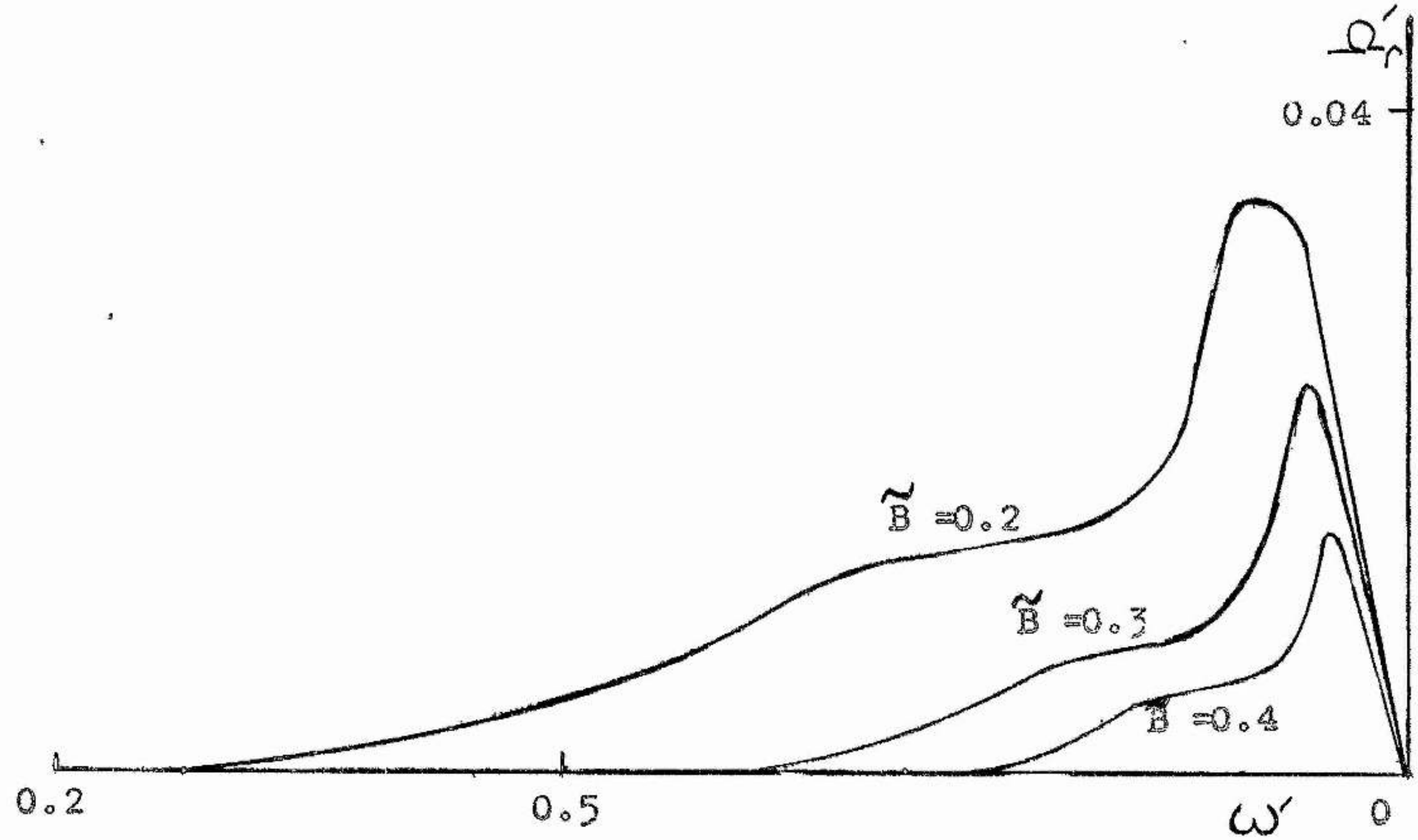


Fig 26

Graph of Ω'_r v ω' for some values of $\tilde{B} > 3/16$.

Table 5

Table of eigenvalues Ω_i' for equation (7.7) when $\omega' = 0$ for some values of B.

B	Ω_i'
0.0	3.95
0.050	1.67
0.075	1.11
0.10	0.745
0.15	0.285
0.15375	0.255

Table 6

Table of the critical values ω_c', $\bar{\omega}_c'$ for various values of B. To within numerical error these values are identical.

B	ω_c'	$\bar{\omega}_c'$
.05	.67 $\pm$ .02	.68 $\pm$ .03
.075	.51 $\pm$ .01	.52 $\pm$ .02
.1	.39 $\pm$ .01	.39 $\pm$ .01
.15	.19 $\pm$ .01	.2 $\pm$ .02
.15375	.18 $\pm$ .01	.19 $\pm$ .02

8. STABILITY OF GENERALISED FIELD BY DIRECT PERTURBATION METHODS

8.1 Introduction

The results of chapter 7 using first-order perturbation theory imply that for certain values of ω' for $\tilde{B} > 0$, it should be possible to find nodeless time-dependent particle-like solutions which are stable. It is necessary to examine this possibility closer using direct perturbation methods. The approach resembles that of chapter 4.

The transformations $\underset{\sim}{\rho} = K\underset{\sim}{r}$, $\tau = Kct$, $\Psi = \frac{\mu}{K}\psi$ allow one to reduce (6.3) to

$$\nabla_\rho^2 \Psi - \frac{\partial^2 \Psi}{\partial \tau^2} = \Psi - \Psi\Psi^*\Psi + \tilde{B}\,\Psi\Psi^*\Psi\Psi^*\Psi \tag{8.1}$$

where $\tilde{B} = \lambda K^2/\mu^4$. The undisturbed state Ψ_0 is again related to the state φ_0' by

$$\Psi_0 = \frac{\mu\,\varphi_0\, e^{-i\omega t}}{K} = \sqrt{1-\omega'^2}\;\varphi_0'\, e^{-i\omega'\tau} \tag{8.2}$$

where φ_0' is the nodeless solution of (6.5)

$$\frac{d^2\varphi'}{dr'^2} + \frac{2}{r'}\frac{d\varphi'}{dr'} = \varphi' - \varphi'^3 + B\,\varphi'^5 \tag{6.5}$$

Equation (8.1) involves the parameter $\tilde{B}$, whereas (6.5) involves the parameter B. These are related by

$$\tilde{B} = \frac{B}{(1-\omega'^2)} \tag{8.3}$$

One/

One can express the energy density in the form

$$\mathcal{E} = \frac{K^4}{\mu^2}\left\{\left|\frac{\partial\Psi}{\partial\tau}\right|^2 + |\nabla_\rho\Psi|^2 + |\Psi|^2 - \tfrac{1}{2}|\Psi|^4 + \frac{\tilde{B}}{3}|\Psi|^6\right\} \tag{8.4}$$

As before we will define the reduced energy density $\mathcal{E}'$ as

$$\mathcal{E}' = \mathcal{E}\mu^2/K^4$$

Our interest lies particularly in particle-like solutions for which the energy density is positive definite. $\mathcal{E}$ will be positive definite if

$$|\Psi|^2 - \tfrac{1}{2}|\Psi|^4 + \frac{\tilde{B}}{3}|\Psi|^6 > 0 \quad \text{for all } \Psi \tag{8.5}$$

$$\text{i.e. } \tilde{B} > 3/16 .$$

Because particle-like solutions exist only for B $[= \tilde{B}(1 - \omega'^2)] < 3/16$, a value of $\tilde{B} > 3/16$ can only be obtained if

$$\omega'^2 > \left(1 - \frac{3}{16\tilde{B}}\right) \tag{8.6}$$

This means that states with positive definite energy density can only occur if the particle-like solution is time-dependent. This immediately raises the question "Could stability be related to $\mathcal{E}$ being positive definite?" The answer is no. It will be shown that states with positive definite energy density are stable, but that stable states for which the energy density is not positive definite can also exist. A graph of the $(\tilde{B}, \omega')$ space is given in fig 27. The regions where

(a) particle-like solutions exist

(b) solutions for which $\mathcal{E}$ must be positive definite

(c)/

(c) solutions where $\mathcal{E}$ need not be positive definite are marked on this graph. Also marked is the curve of critical $\tilde{\omega}'_c$ as a function of $\tilde{B}$, obtained in chapter 7. For $\omega' > \tilde{\omega}'_c$ first-order perturbation theory implies stability. This in turn implies that stable particle-like solutions can exist for which $\mathcal{E}$ is not positive definite. For example the point $\tilde{B} = 5/32$ $\omega' = 0.6$ $B = 0.1$ is, from first-order perturbation theory a stable point but it does not lie in the region in which $\mathcal{E}$ is positive definite since $\tilde{B}$ is less than $3/16$.

It might be argued that we have been too severe in requesting that $\mathcal{E}$ remain positive definite for all disturbances. After all, section 7 deals with small disturbances, and thus the condition of stability to small disturbances might not be that $\mathcal{E}$ be positive definite to all disturbances but only so to small disturbances. It can again be shown that this hypothesis is not likely to be correct.

Let us consider the case $B = 0.1$. Then $\mathcal{E}$ as given by (6.6) becomes positive definite when

$$\omega' > \sqrt{\frac{3 - 16B}{3 + 16B}} = 0.55.$$

But if one refers back to chapter 7, one sees that the value of ω'_c for $B = 0.1$ is $.39 \pm .01$. This means that according to first-order perturbation theory we can have stable solutions for which not even the undisturbed particle-like solution need have an energy density which is everywhere positive. Later, in chapter 9 we suggest a criterion for stability, which is related not to the energy density, but to the total integrated energy. We now verify some of the results of chapter 7 by direct perturbation methods.

8.2 Examination of the stability of particle-like solutions

Examination of fig 27 shows that the $(\tilde{B}, \omega')$ plane can be split into 4 distinct regions.

(i) Region for which no particle-like solutions exist (i.e. $B > 3/16$)

(ii) Region in which particle-like solutions exist for which the energy density is unconditionally positive definite ($\tilde{B} > 3/16$). The results of first-order perturbation theory imply the existence of stable solutions.

(iii) Region in which particle-like solutions exist for which $\mathcal{E}'$ need not be positive definite but which are predicted to be stable from the results of first-order perturbation theory.

(iv) Region in which the results of first-order perturbation theory predict that solutions are unstable

Let us denote typical points in the regions (ii), (iii), (iv) by S, R, U respectively.

When a 'particle' is unstable, then a convenient manner of following its decay and so discussing its stability and decay modes, is to plot the reduced energy density $\mathcal{E}'$ against the reduced radial distance ρ for a series of values of reduced time τ. This was the method used to illustrate the singular and dissipative decays in the $\tilde{B} = 0$ case discussed in chapter 4. If the 'particle' is stable, then this method is not satisfactory, and an alternative method of representation is adopted. It was found preferable to plot the energy and the charge within given radii as a function of τ.

We/

We define

$$\bar{Q}(a) = \frac{1}{4\pi}\int_0^a Q_{dens}\, d^3r' \tag{8.7}$$

$$\bar{E}(a) = \frac{1}{4\pi}\int_0^a \mathcal{E}'\, d^3r' \tag{8.8}$$

where Q_{dens} is given by

$$Q_{dens} = -i\left(\Psi \frac{\partial \Psi^*}{\partial \tau} - \Psi^* \frac{\partial \Psi}{\partial \tau}\right)$$

8.3 Stability of S

For illustration purposes we take a definite point S in region (ii). For this discussion S is taken to be the point $\tilde{B} = 5/18$, $\omega = 0.8$. In this section, by a series of graphs, we will illustrate the effect of various types of disturbance on S, showing how stable S really is. A disturbance to which S is unstable will also be given. Thus, although S can be destroyed, it will be shown to be capable of withstanding extremely severe disturbances. In sections 8.4 and 8.5 we consider the stability of points R and U though in less detail. A discussion of the results is postponed until chapter 9.

Random disturbances similar to those applied to the B = 0 case were applied to S. To all such disturbances, S was completely stable. For an undisturbed state, a graph of $\bar{E}$ (a), $\bar{Q}$ (a) v τ for various a would consist of a series of straignt lines parallel to the τ axis. When/

When S was disturbed by random disturbances, the behaviour so closely approximated this monotonous form, that no graphs of this are given. Instead we consider more interesting phenomena

Fig 28 is an example of what happens when the solution for S is stretched and then let go. $\bar{E}(a)$, $\bar{Q}(a)$ are plotted against τ for the values a = 2, 4, 6, 8, except that $\bar{Q}$ (8) is not drawn because it is too close to $\bar{Q}$ (6) to resolve clearly. The stretching corresponds to disturbing the particle to the state

$$\Psi\Big|_{\tau=0} = 1.25\,\Psi_{os}\Big|_{\tau=0} \qquad \frac{\partial\Psi}{\partial\tau}\Big|_{\tau=0} = 1.25\,\frac{\partial\Psi_{os}}{\partial\tau}\Big|_{\tau=0}$$

where Ψ_{os} is the undisturbed solution for S. As can be seen from the graph, the particle radiates only some of the excess energy it acquired from the stretching but returns to an undisturbed state. There is some evidence of oscillation in the $\bar{E}(a)$, $\bar{Q}(a)$ levels. For example, $\bar{E}(6)$ can be seen to have a long period oscillation and $\bar{E}(2)$ to have a short period oscillation.

Some attempts were made to force the particle to oscillate at an unnatural frequency. An example of this is given below. The disturbed state is defined by

$$\Psi\Big|_{\tau=0} = 0.6\,\varphi_0' e^{-i0.5\tau}\Big|_{\tau=0} \qquad \frac{\partial\Psi}{\partial\tau}\Big|_{\tau=0} = -i0.3\,\varphi_0' e^{-i0.5\tau}\Big|_{\tau=0}$$

The natural angular frequency of oscillation is $\omega' = 0.8$, and we have thus/

thus decreased this frequency from the natural value to an unnatural value of 0.5. The result of so doing is shown in fig 29 where $\bar{E}(2)$, $\bar{E}(4)$, $\bar{E}(6)$, $\bar{Q}(2)$, $\bar{Q}(4)$ are plotted as functions of τ. There are several points to note here. In fig 28 oscillations in the $\bar{E}$, $\bar{Q}$ levels were only just discernable, but here they are clearly visible on both the charge and energy levels. Although $\bar{E}(\infty)$ is not actually plotted, it is only marginally larger than $\bar{E}(6)$ since effectively all of the energy is contained within a sphere of radius $\rho = 6$. This means that $\bar{E}(\infty)$ for this case is considerably less than $\bar{E}(\infty)$ for the undisturbed state, yet there is no decay. (For the undisturbed state S, the value of $\bar{E}(\infty)$ is 38.0) Some energy is radiated as can be seen from $\bar{E}(6)$, but if one integrates for a longer time than shown on the graph, $\bar{E}(6)$ levels out. Further, although the oscillations are of fixed period, this does not represent a steady state. The amplitude of the oscillations is decreasing and the magnitudes of $\bar{E}(2)$, $\bar{Q}(2)$ show signs of increasing slowly. This probably represents an attempt of ψ to return to some equilibrium position, though the return is slow and cannot be to the initial undisturbed state. This will be discussed further in chapter 9. From fig 29 the angular frequency of the oscillations can be estimated to be $\sim$ 1.6 radians/unit of τ

Disturbances, initiating outside the particle radius, were tried. These disturbances are similar in behaviour to that shown in fig 11 in chapter 4 except that in this case the particle does not decay. Disturbances oscillating at a frequency $\omega' > 1$ were also tried in an attempt/

attempt to induce a decay but without success. Some of the disturbances falling into the above categories can be seen from the graphs 31 - 35 to be extremely severe yet the particle has no tendency to decay, though it usually radiates some energy before settling down.

Fig 30. The disturbance applied is

$$\Psi_1\Big|_{\tau=0} = 0{\cdot}01\,\rho\,\varphi_0'^{\,2} \qquad \frac{\partial\Psi_1}{\partial\tau}\Big|_{\tau=0} = 0{\cdot}03\,\rho\,\varphi_0'^{\,2}$$

This disturbance causes no great change in the energy configuration of the particle, though there is an initial period of adjustment, and there is oscillation, clearly visible on the levels $\bar{\bar{E}}(2)$, $\bar{\bar{Q}}(2)$, $\bar{E}(4)$. The angular frequency of this oscillation is ~ 1.6 radians/unit of τ.

Fig 31. The disturbance in this case, is defined by

$$\Psi_1\Big|_{\tau=0} = \mathcal{G}_1(10) \qquad \frac{\partial\Psi_1}{\partial\tau}\Big|_{\tau=0} = -3\,\mathcal{G}_1(10)$$

where $\mathcal{G}_1(10)$ is defined in Appendix M.

One can see the disturbance travelling in and colliding with the particle. It enters the radius $\rho = 6$ at $\tau \sim 4$ units, the radius $\rho = 4$ at $\tau \sim 7$ units, $\rho = 2$ at $\tau \sim 10$ units. There is also some evidence of the disturbance travelling outwards after exciting the particle but this is less well defined because the particle is now in an excited state. After excitation, there is a period of readjustment and then a return to a more steady state which can be seen to be more or less the same as the initial unperturbed state. There is also some evidence/

of oscillation particularly on $\bar{\bar{E}}(4)$, but the amplitude of this is small.

Fig 32. In this case the disturbance is of the same form as for fig 31 but is applied nearer the particle at $\rho = 4$. Thus it has already entered the radii $\rho = 10, 6$, but it can be clearly seen entering the radius $\rho = 2$, travelling in to $\rho = 0$ and then propagating outward through the now wholly excited particle. This disturbance is fairly violent causing marked oscillation. A considerable amount of excess energy is radiated as can be seen by the rapid fall of $\bar{\bar{E}}(10)$ and $\bar{E}(6)$. The angular frequency of the oscillations is ~ 1.6 radians/units of τ

Fig 33. This gives a clear indication of a disturbance applied well outside the particle-radius travelling in to $\rho = 0$ and then propagating outwards again leaving the particle in an excited state. It quickly returns to a more or less equilibrium state, however, exhibiting some long period oscillations as it does so. An estimate of the angular frequency of these slow oscillations is ~ 0.2 radians/ τ - unit.

Fig 34. This disturbance is effectively two disturbances, both outside the particle-radius, but one further out than the other. Initially, $\bar{E}(20)$ contains one disturbance centre but not the other, which can be seen entering the radius $\rho = 20$. Initially both disturbances lie outside $\rho = 6$. The first one can be seen entering but the entrance of the second is masked because of the excitation caused by the first. One effect of this double disturbance is to produce large oscillations of long period. Even in the time interval shown, the amplitude of these decreases, and if one integrates for a longer time than shown in the figure/

figure, a return to an unexcited state close to the initial undisturbed state results. An estimate of the angular frequency of the slow oscillations is $\sim$ 0.25 radians/ τ - unit.

This disturbance is defined by

$$\Psi_1\Big|_{\tau=0} = 20.\, \mathcal{G}_2(15) + i\, 20.\, \mathcal{G}_2(22)$$

$$\frac{\partial \Psi_1}{\partial \tau}\Big|_{\tau=0} = 5.\, \mathcal{G}_2(15) - 60.\, \mathcal{G}_2(22)$$

Fig 35. When the disturbance given by

$$\Psi_1\Big|_{\tau=0} = 2\, \mathcal{G}_1(10) \qquad\qquad \frac{\partial \Psi_1}{\partial \tau}\Big|_{\tau=0} = i\, 8\, \mathcal{G}_1(10)$$

is applied, it has the effect on the particle shown in fig 35. Again the disturbance can be clearly seen entering ρ = 10, 8, 6, 4, 2 and can also in a sense be seen leaving, particularly so for the radii ρ = 6, 8, 10. The particle is provoked by the disturbance into producing both long and short period oscillations, the short period ones being most visible on $\bar{E}(2)$, $\bar{Q}(2)$, $\bar{\bar{E}}(4)$, and the long period ones on $\bar{Q}(4)$, $\bar{Q}(6)$, $\bar{E}(4)$, $\bar{\bar{E}}(6)$, $\bar{\bar{E}}(8)$, $\bar{\bar{E}}(10)$.

The long period oscillations have an estimated angular frequency of 0.22 radians/τ-unit. For the shorter period oscillations the appropriate value is $\sim$ 1.5 radians/unit of τ.

So far a variety of disturbances have been applied to S, but the particle/

particle has never responded by decaying. It has usually oscillated, and radiated some energy, but there has never been any attempt at a decay. A disturbance is now given which induces a dissipative decay.

We investigated the behaviour of S to the disturbance

$$\Psi\Big|_{\tau=0}, \quad \frac{\partial \Psi}{\partial \tau}\Big|_{\tau=0} \quad \text{where} \quad \Psi \quad \text{has the form}$$

$$\Psi = \bar{\varphi}' e^{-i0.8\tau} + 0.01\,\rho\,\bar{\varphi}'^{2} e^{-i3\tau}$$

and $\bar{\varphi}'$ is the spatial part of the unperturbed state for the values of the parameters $\tilde{B} = 0.1$, $\omega' = 0$. The effect of applying this disturbance can be seen in fig 36. The particle responds by rapidly decaying i.e. radiating all its energy! This is an example of a dissipative decay. Although this shows that S is not unconditionally stable, it is stressed that it is nevertheless exceedingly stable being capable of withstanding extremely severe disturbances.

Conclusion

The particle-like solution for the point S has $\mathcal{E}'$ positive definite and is stable to extremely severe disturbances, as well as to small disturbances. In general, when one applies a large disturbance the particle is excited and usually provoked into radiating some excess energy before returning to a calmer state. During this 'calming down' period there is a tendency for the particle to show oscillations. Two types of oscillation occur, a long period low frequency oscillation, and a higher frequency oscillation. In general the angular frequency of/

of the low frequency oscillation is of the order of 0.2 radians/unit of τ while for the high frequency oscillation this value is $\sim$ 1.6 radians/ τ-unit.

8.4 Stability of R.

Examination of the stability of R reveals that the results of first-order perturbation theory are correct and that such points really are stable. We give a few examples, taking as a typical point, the one defined by $\tilde{B} = 5/32$, $\omega' = 0.6$.

Fig 37 The disturbance given by

$$\Psi_1\Big|_{\tau=0} = -\mathcal{G}_1(4) - i\,\mathcal{G}_1(6) \qquad \frac{\partial\Psi_1}{\partial\tau}\Big|_{\tau=0} = 3\,\mathcal{G}_1(4)$$

is applied. This is a disturbance applied outside the particle radius. The behaviour is similar to that exhibited by S: there is a period of excitation followed by a return to a calmer behaviour, with evidence of damped oscillations, the angular frequency of which is estimated to be $\sim$ 0.65 radians/ τ- unit.

Fig 38. The disturbance in this case is extremely severe yet the particle survives though it does radiate some excess energy. This example shows how drastic a disturbance such a state can withstand. Oscillation and a return to a less excited state are again manifest. The disturbance is defined by

$$\Psi_1\Big|_{\tau=0} = -5\,\mathcal{G}_1(4)\,e^{i3\tau}\Big|_{\tau=0}$$

$$\left.\frac{\partial \Psi_1}{\partial \tau}\right|_{\tau=0} = -i15\, g_1(4)\, e^{-i3\tau}\Big|_{\tau=0}$$

Fig 39 This corresponds to a stretching of the particle. Large oscillations occur with decaying amplitude. The particle returns to a steady state. An estimate of the angular frequency is ~ 0.6 radians/ τ-unit. In this case it is also possible to estimate the damping factor at $0.035/\tau$-unit. The initial disturbed state is given by

$$\Psi\Big|_{\tau=0} = 1.125\, \Psi_{0R}\Big|_{\tau=0}$$

$$\left.\frac{\partial \Psi}{\partial \tau}\right|_{\tau=0} = 1.125 \left.\frac{\partial \Psi_{0R}}{\partial \tau}\right|_{\tau=0}$$

8.5 Examination of U

This state is similar to the B = 0 case in that it is highly unstable. There is a difference however. The singular decay as described in chapter 4 does not occur, but is replaced by another unphysical decay, which also is a consequence of $\mathcal{E}$ not being positive definite. An example of this decay mode is shown in fig 40. U is taken to be the point $\tilde{B} = 0.1$, $\omega' = 0.2$. The disturbance is a random one similar to that used for the B = 0 case. Instead of going singular, there/

there is a tendency for $\mathcal{E}'$ to be made negative and constant over an increasingly large sphere. The negative energy required to do this is obtained by increasing the energy in the $\mathcal{E}'$ positive region. This was also done in the singular decay mode of the B = 0 case, but then the positive bump did not travel outwards as it must do here. The negative value that $\mathcal{E}'$ wishes to obtain seems to be characteristic for a given $\tilde{B}$, ω'. For example fig 41 shows the effect of stretching the particle. When released, there is an immediate tendency to adopt the above decay. A brief examination of figs 40 and 41 is sufficient to show that both solutions want to attain the constant value $\mathcal{E}' = -7.2$. For different disturbances for this value of $\tilde{B}$ and ω' this same tendency is observed.

8.6 Summary

The results of first order perturbation theory are confirmed by direct perturbation techniques. The criterion for stability is shown not to be that $\mathcal{E}$ be positive definite. The stability of a point for which $\mathcal{E}$ must be positive definite (S), one for which it need not be (R), and one for which it is not (U) are considered. First-order perturbation theory predicts both S and R to be stable. It is found that R and S exhibit very much the same behaviour, that both are in fact extremely stable, though both can be destroyed. U is rather uninteresting because it is so unstable. A consequence of the fact that $\mathcal{E}$ is not positive definite, is that an unphysical decay mode for U is possible, whereby $\mathcal{E}$ assumes/

assumes some negative constant value over a region of space which increases as τ increases. To compensate for the negative energy, $\mathcal{E}$ develops an increasing bump in the positive region which travels outwards.

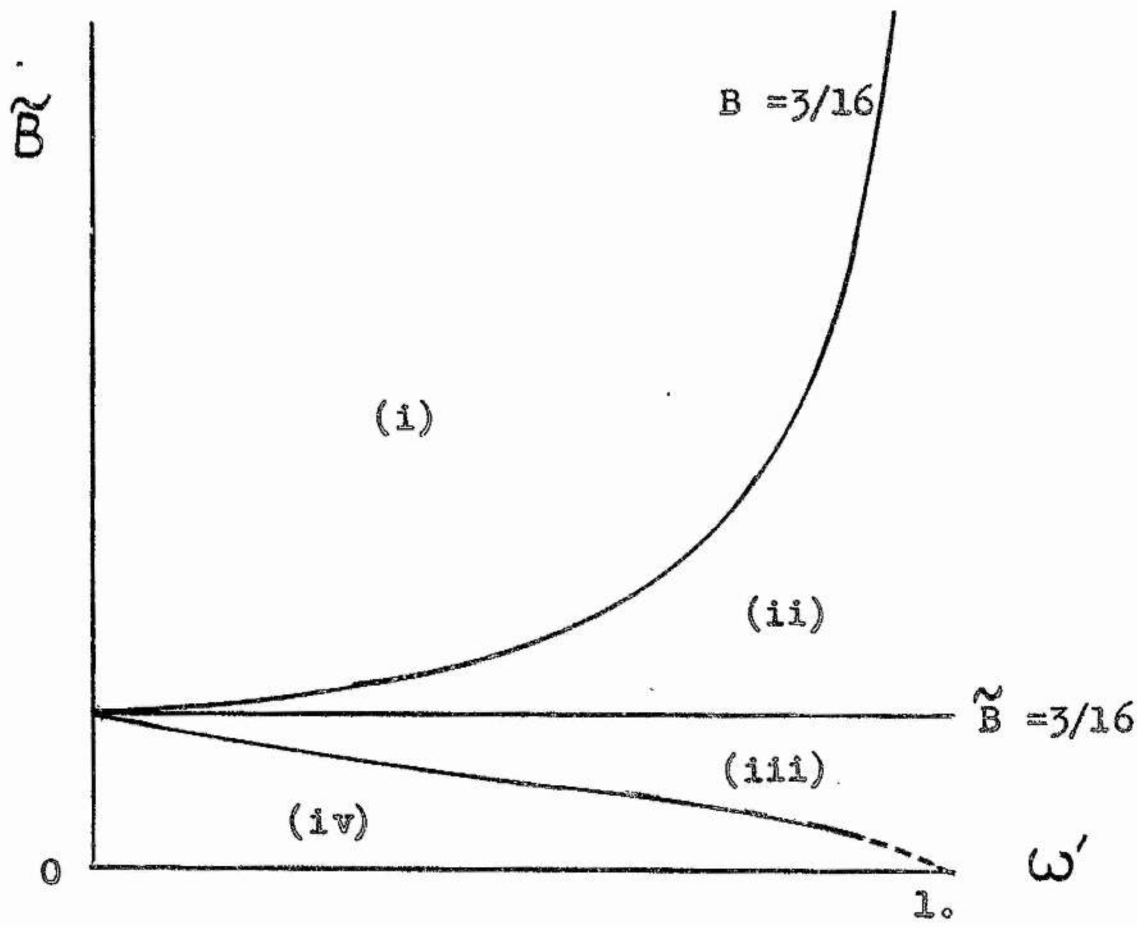


Fig 27

Graph of $\tilde{B}$ v ω' showing how the space splits into four regions.

(i) Region of no particle-like solutions (B $>$ 3/16).

(ii) Region where the energy density is positive definite.

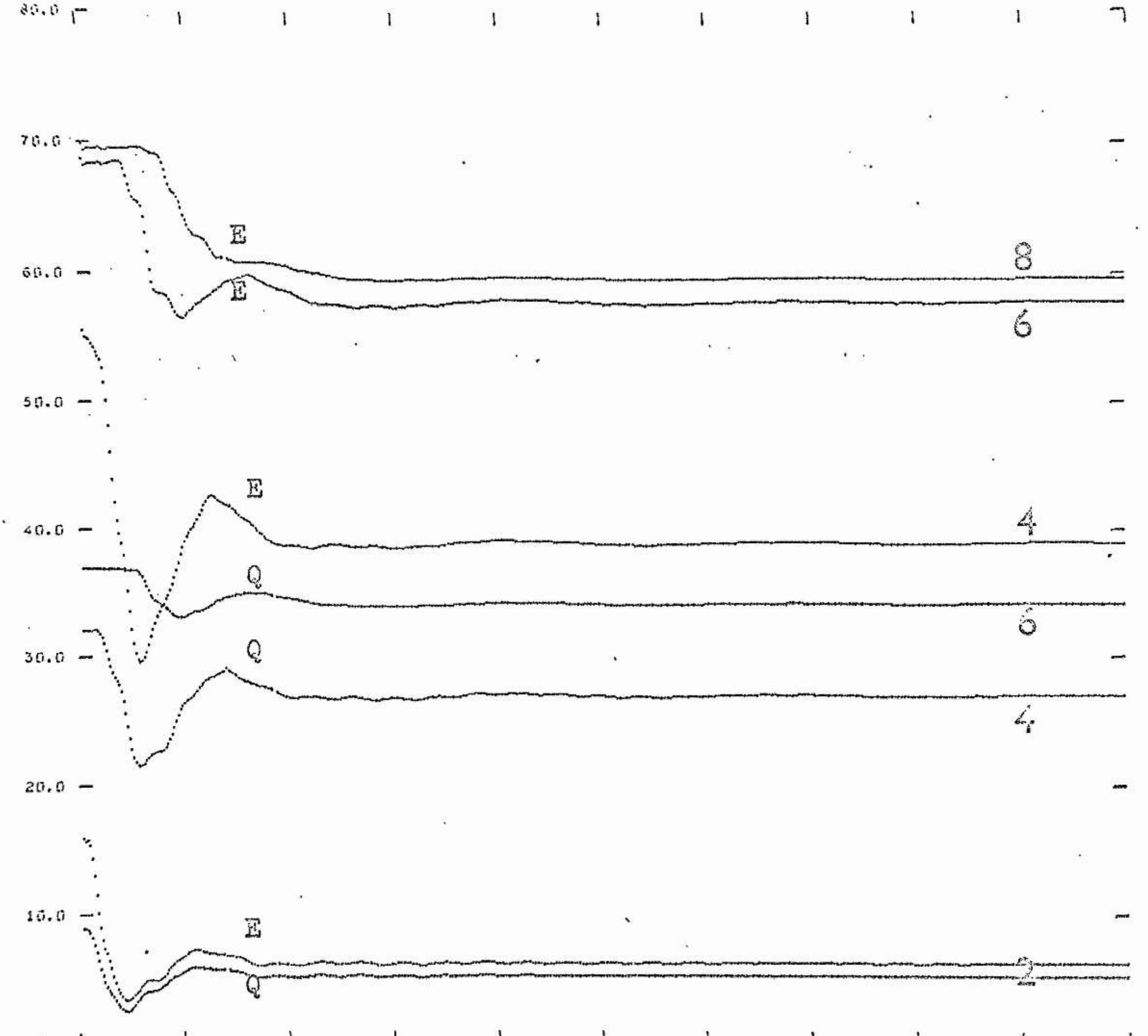
(iii) Region of stable solutions with $\mathcal{E}'$ not +ve definite.

(iv) Region of unstable particle-like solutions.

Figs 28 – 39 inclusive are copies of graphs produced by the SC 4020 graphical output unit of the Science Research Council's Atlas Computer, Chilton. The graphs show the reduced energy $\bar{\bar{E}}(a)$ and the reduced charge $\bar{\bar{Q}}(a)$ plotted against the reduced time τ for various values of a. The letters E, Q at the left-hand side indicate whether the curve is an energy or a charge curve, and the numbers at the right-hand side give the values of a for that curve.

The undisturbed levels for $\bar{\bar{E}}(a)$, $\bar{\bar{Q}}(a)$ for the point S are given below.

a	$\bar{\bar{E}}(a)$	$\bar{\bar{Q}}(a)$
2	6.0	4.9
4	29.9	18.0
6	37.4	20.7
8	38.1	20.94
10	38.2	20.95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8
6
4
6
4
2

Fig 28

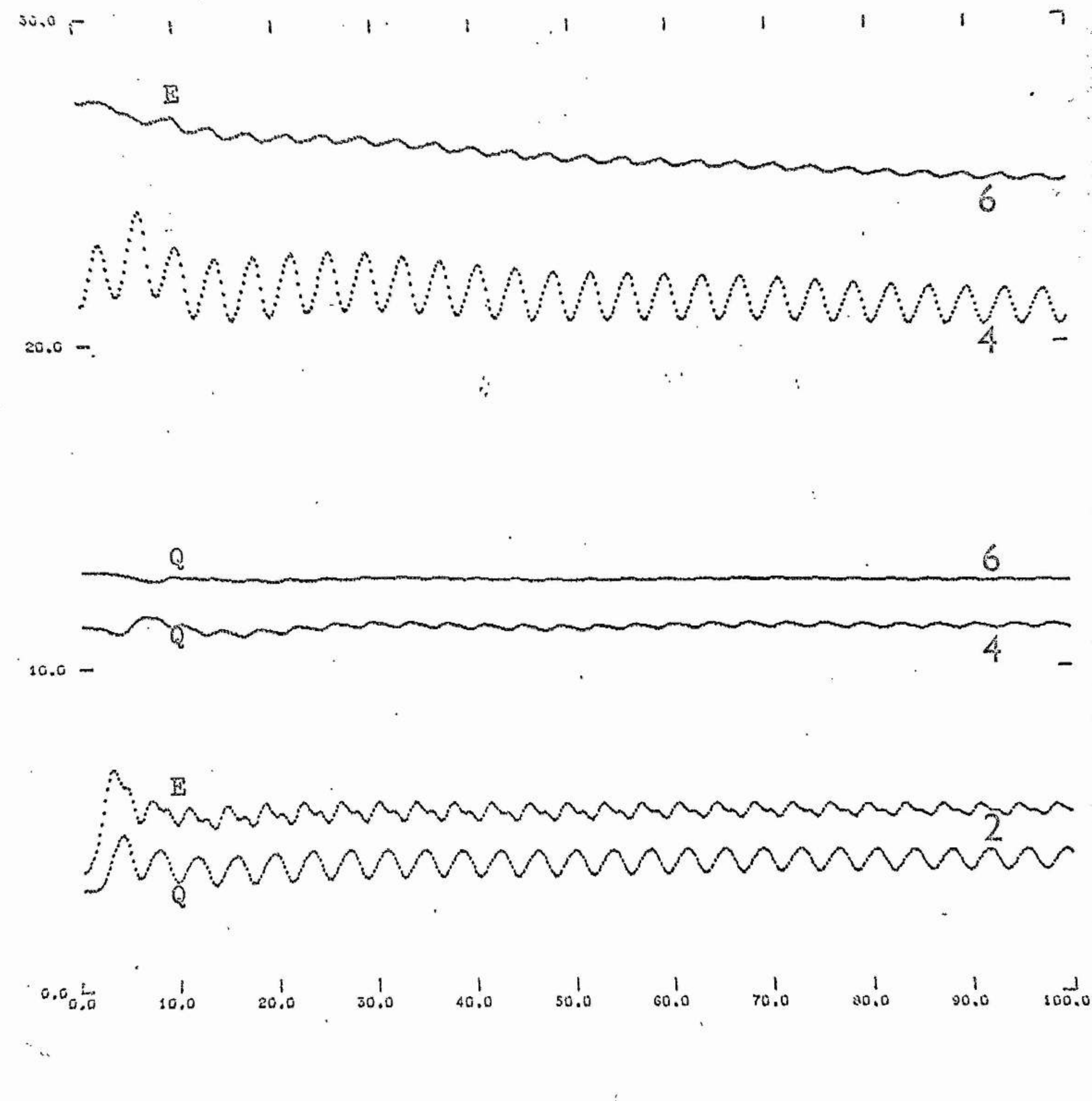
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Fig 29

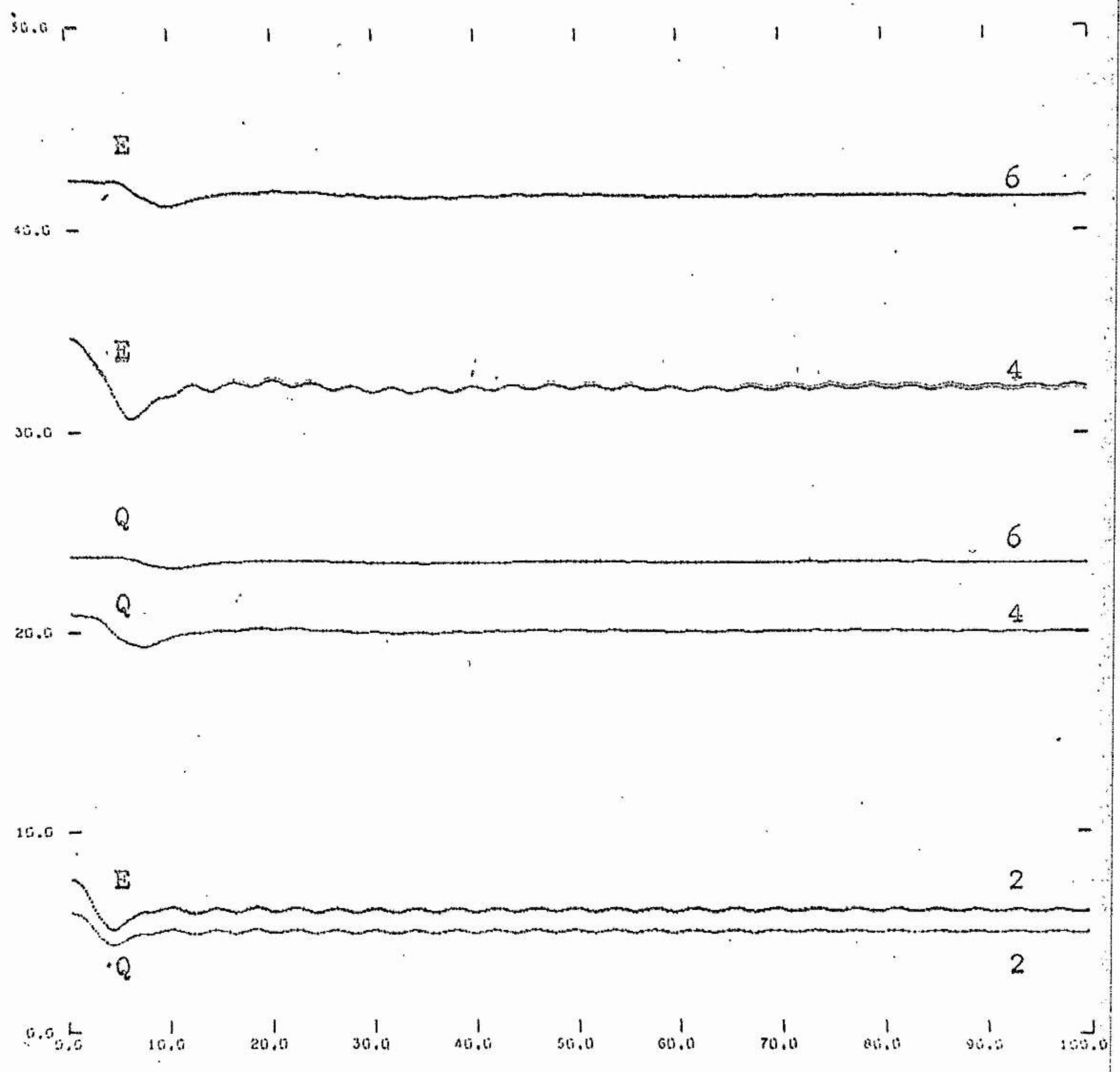
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Fig 30

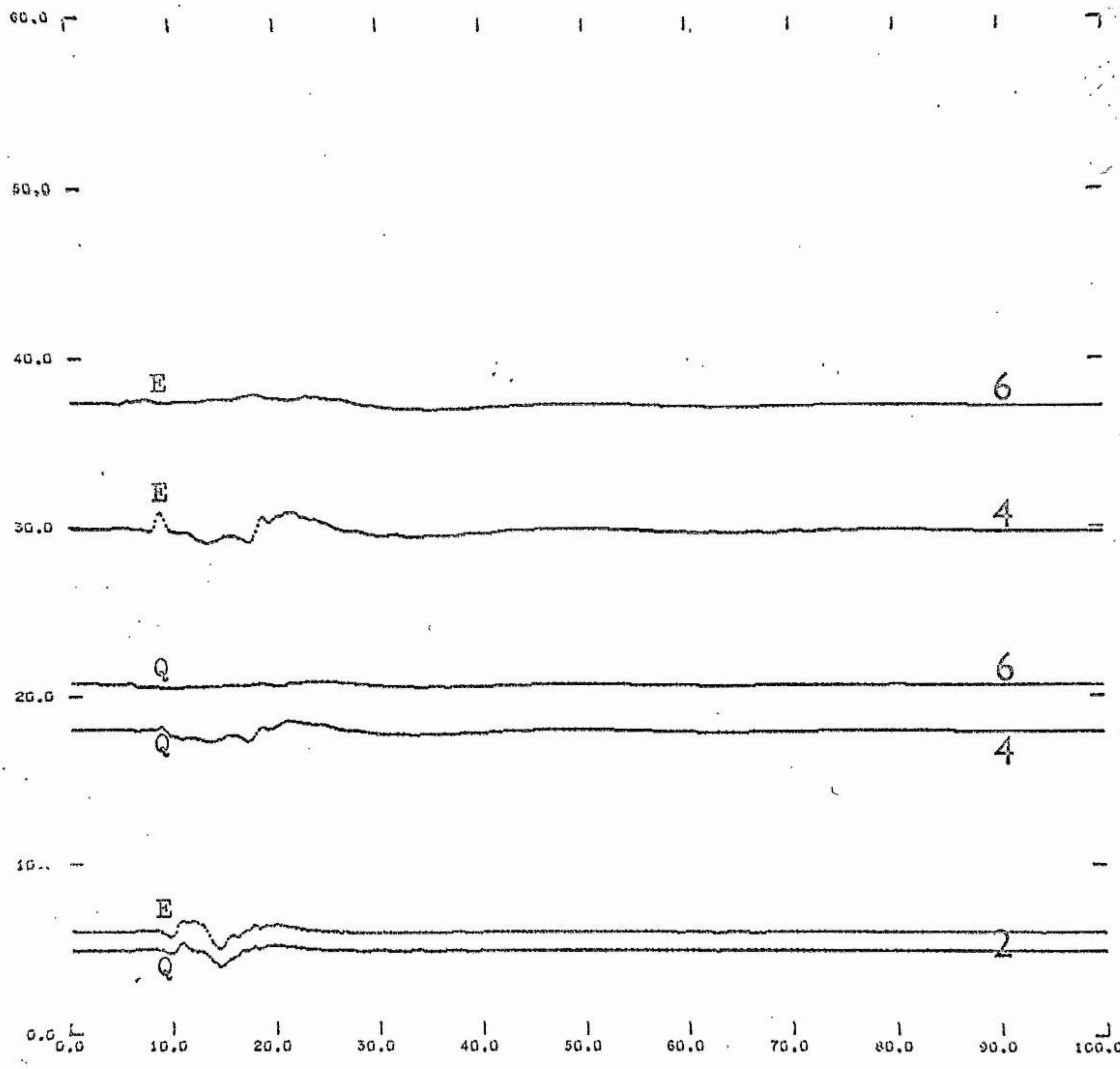
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Fig 31

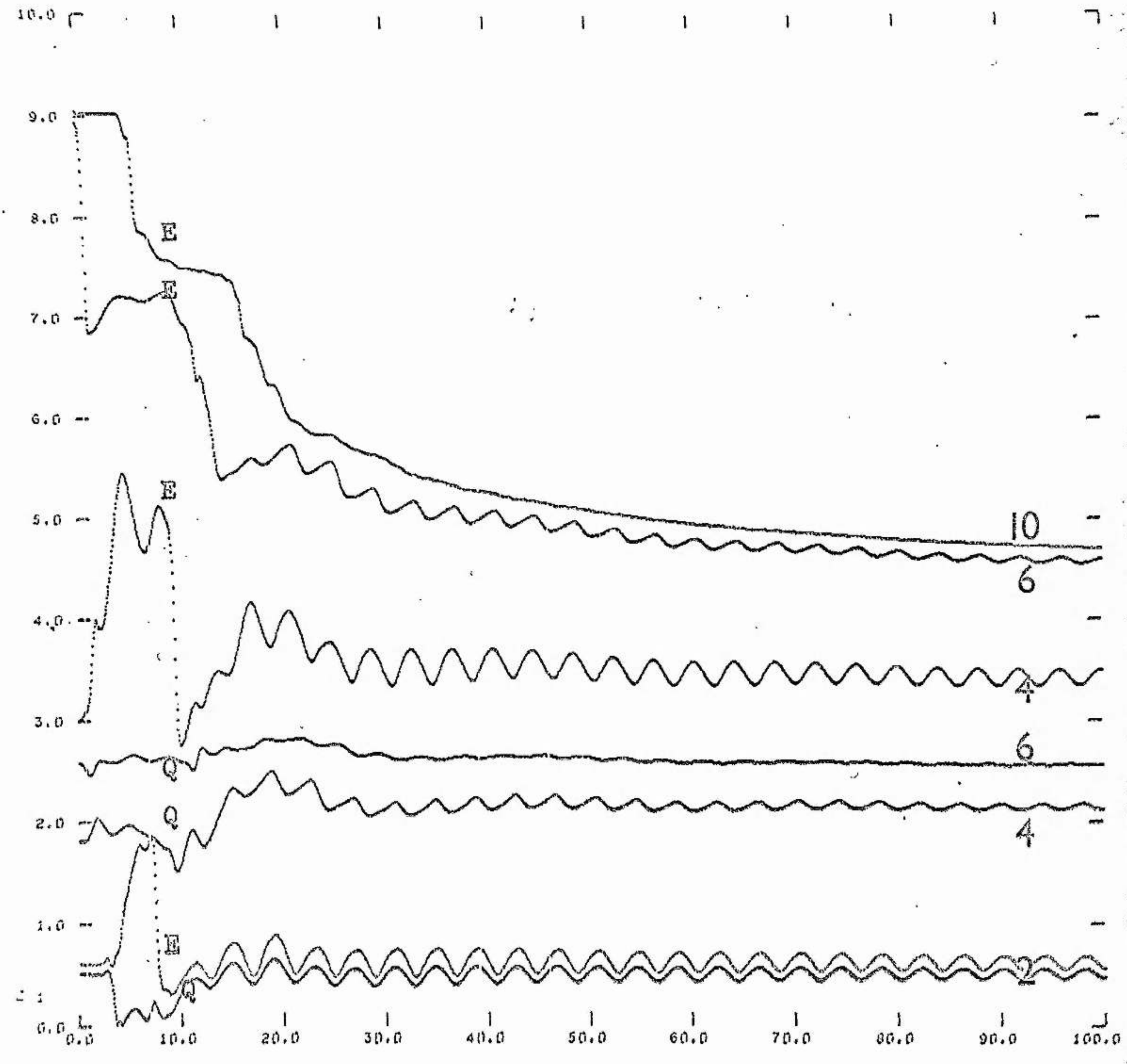
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Fig 32

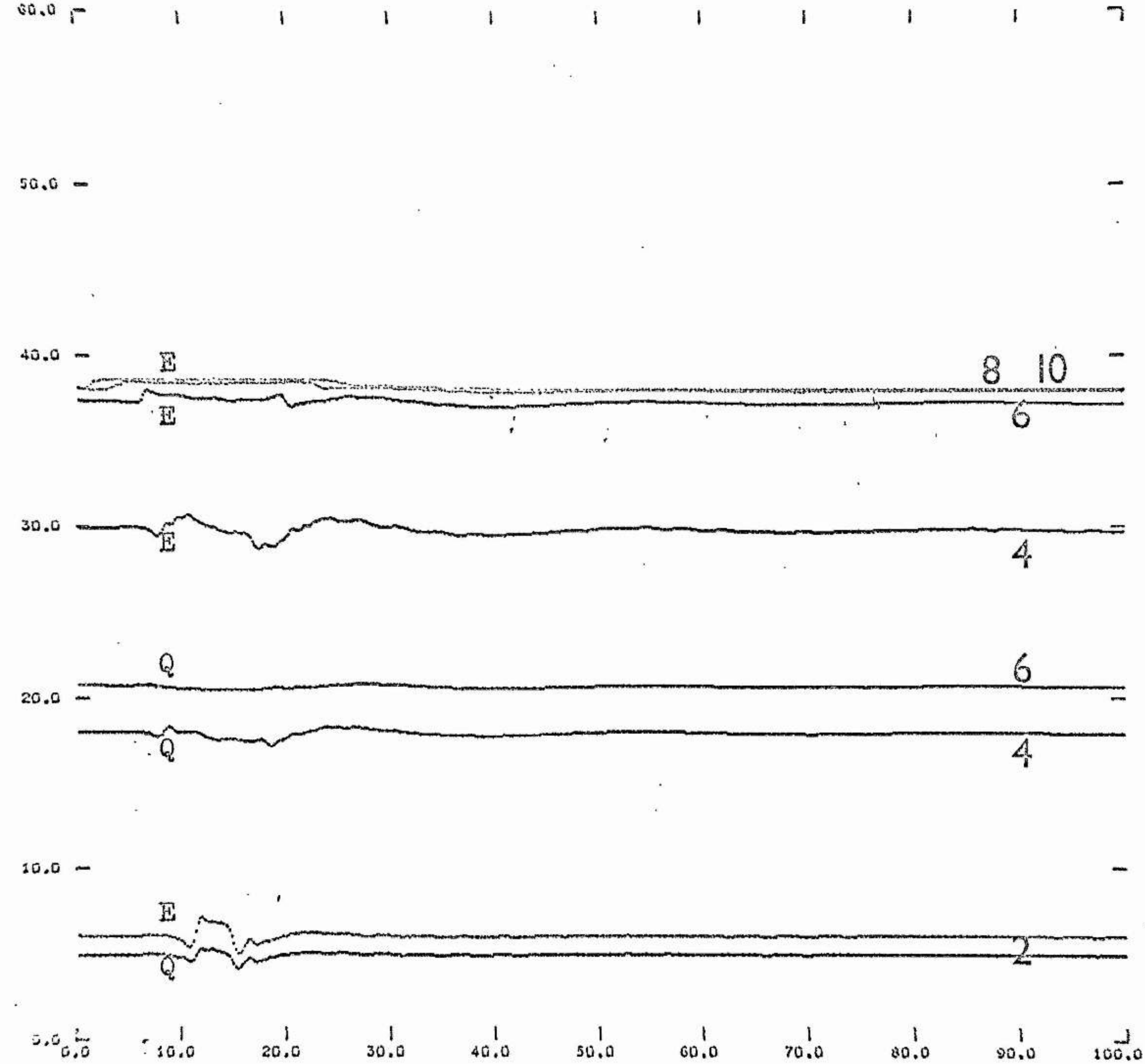
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Fig 33

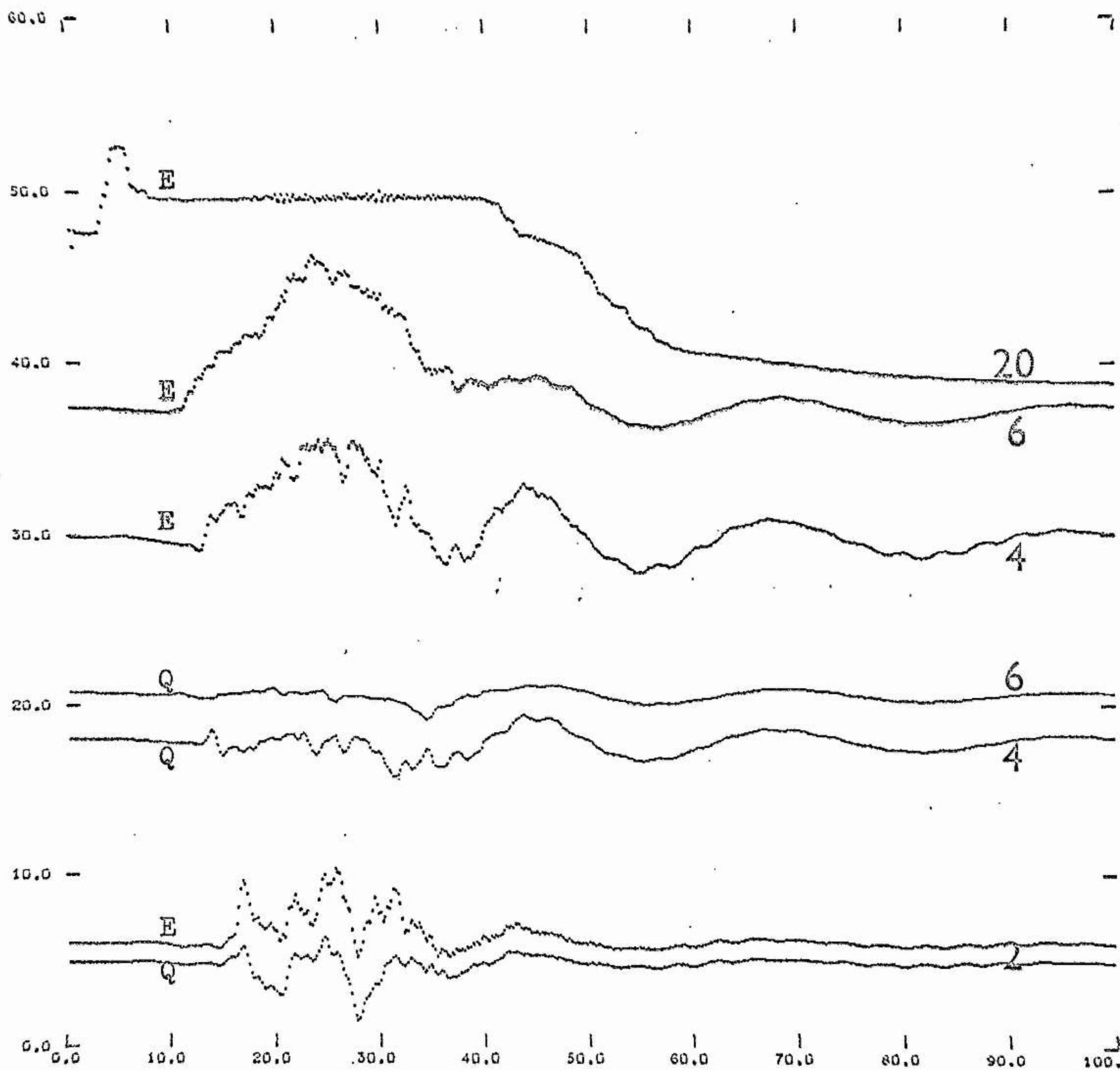
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Fig 34

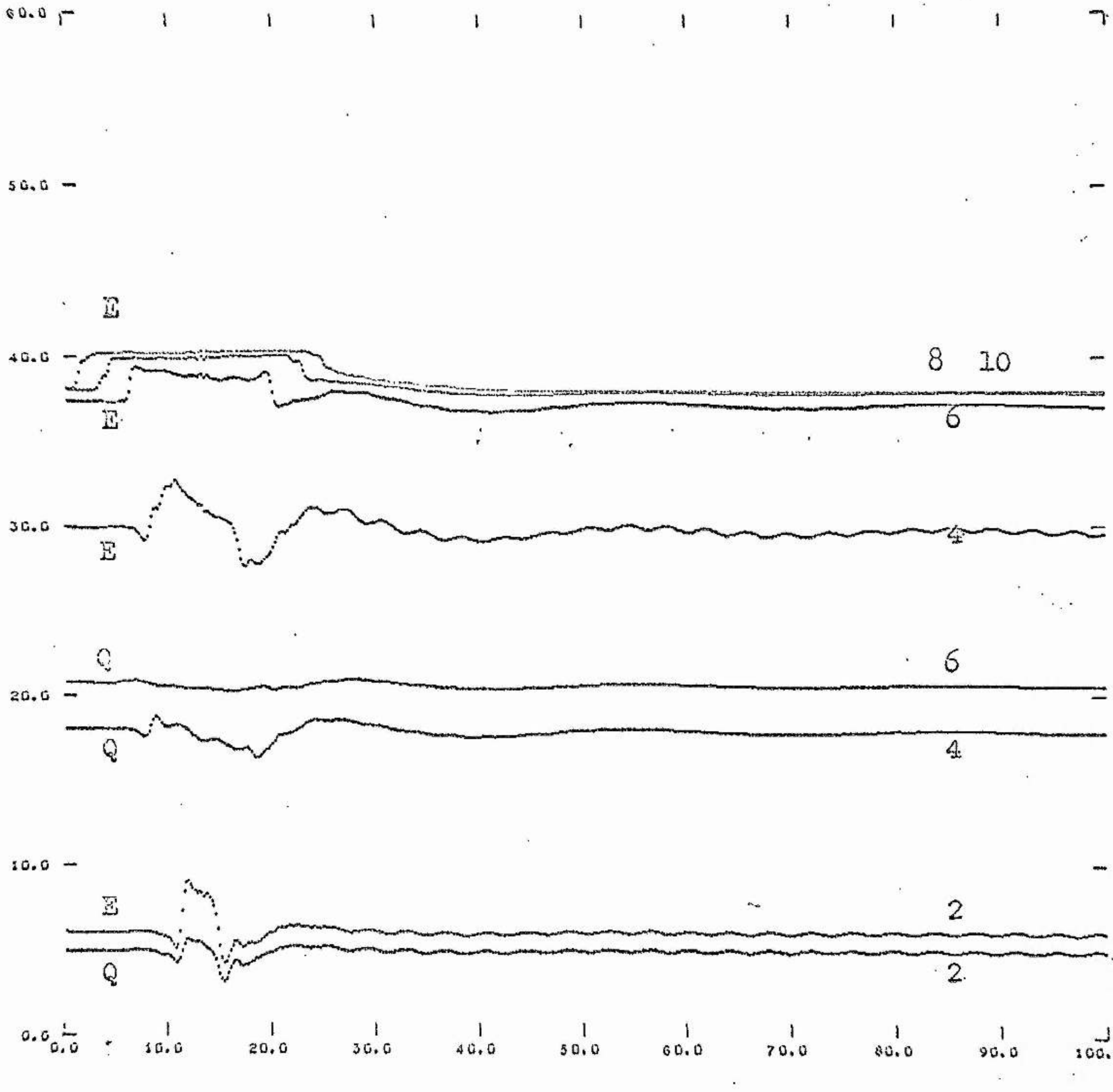
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Fig 35

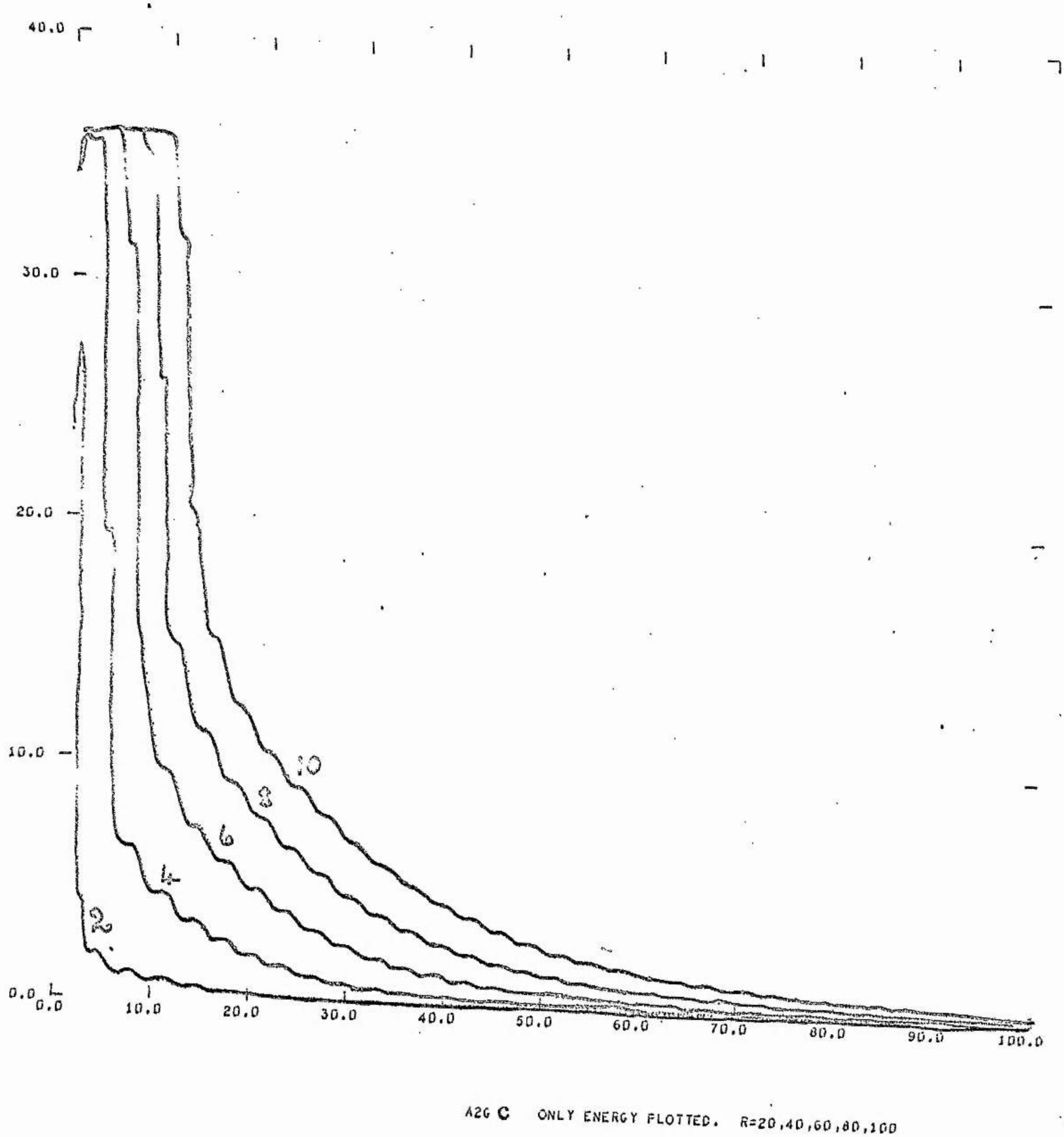

fig 36

Example of a dissipative mode of decay for the point S. $\overline{E}(a)$ is plotted against the reduced time τ for the values of a = 2, 4, 6, 8, 10.

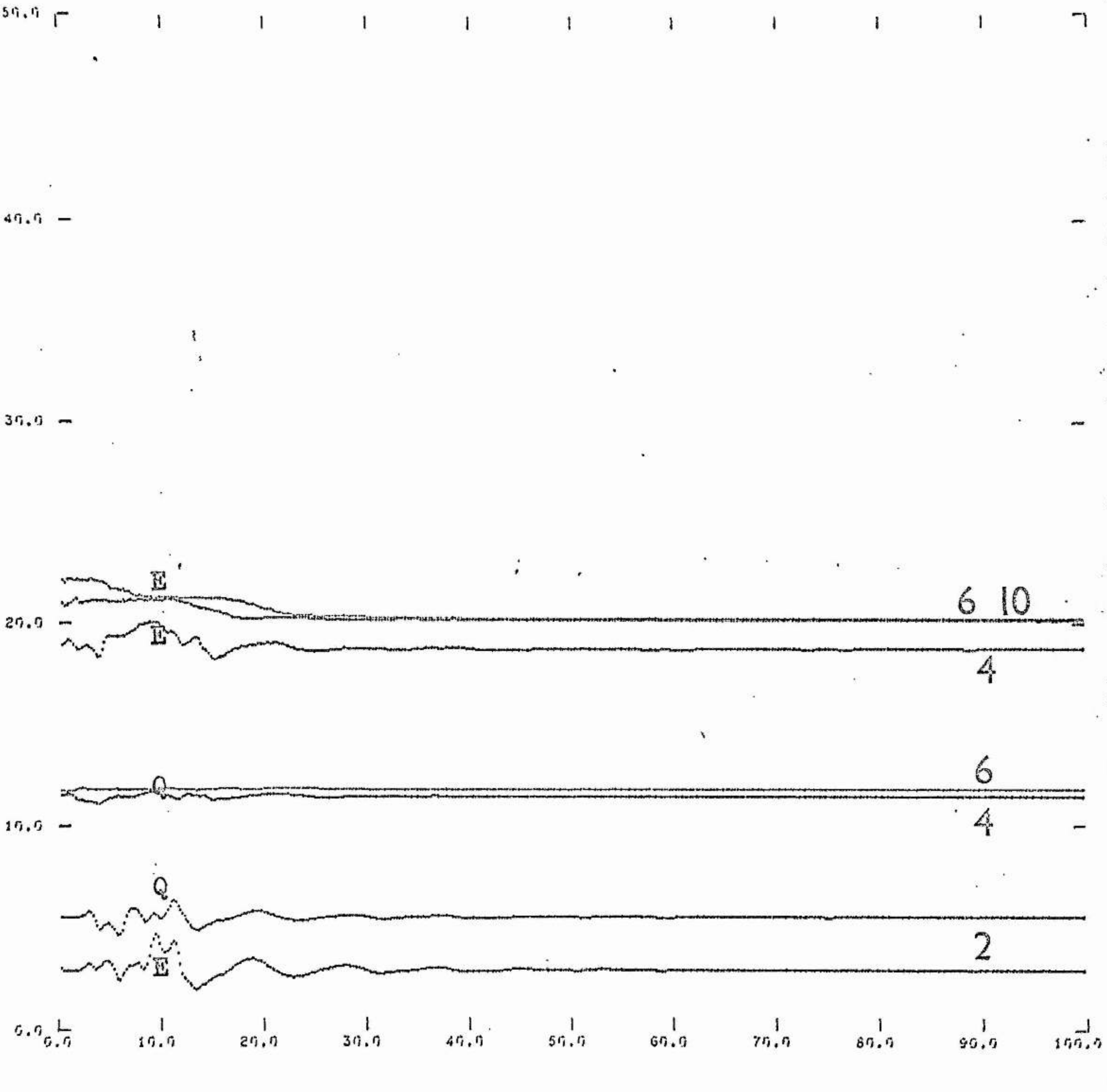
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Fig 37

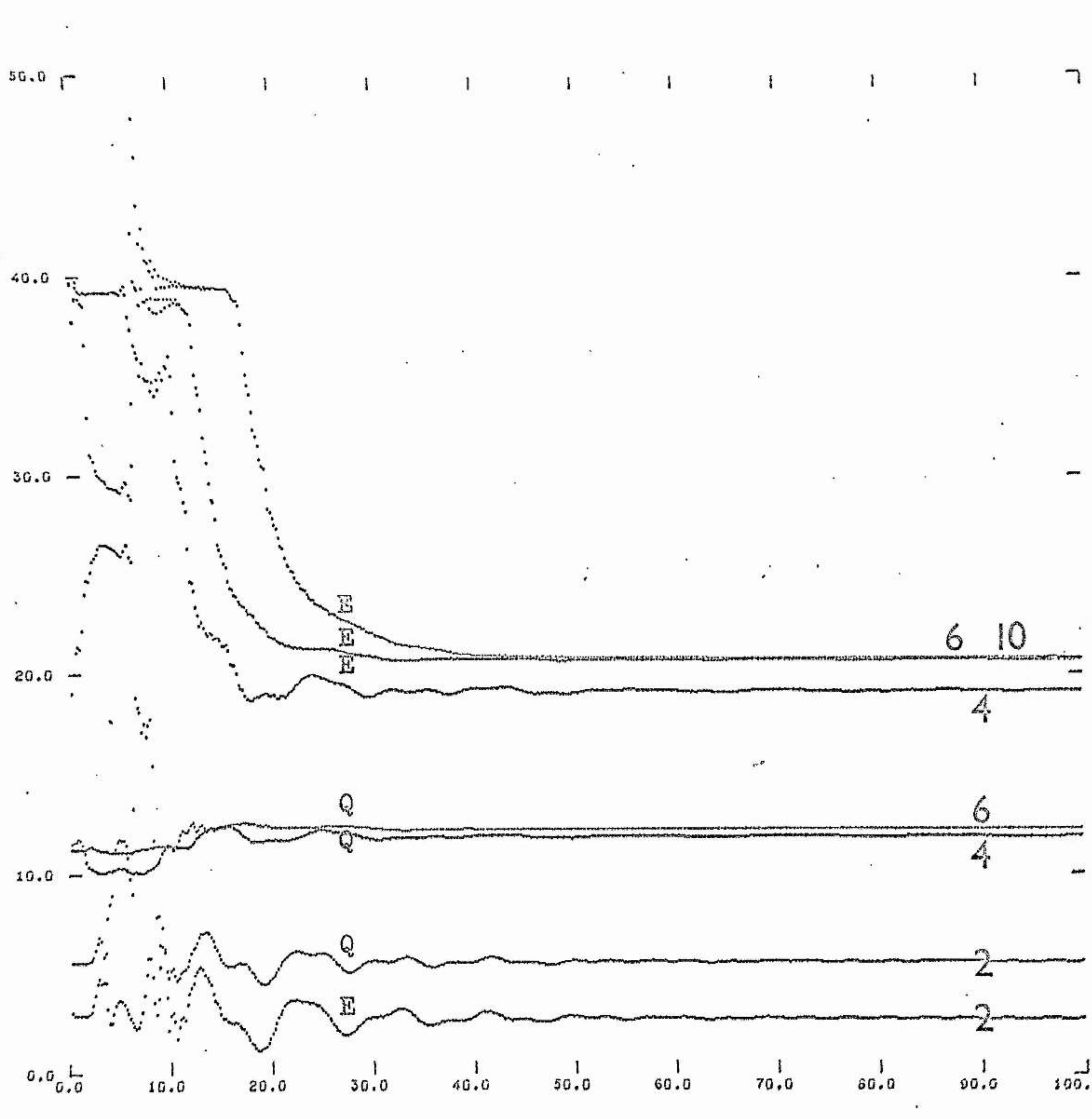
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Fig 38

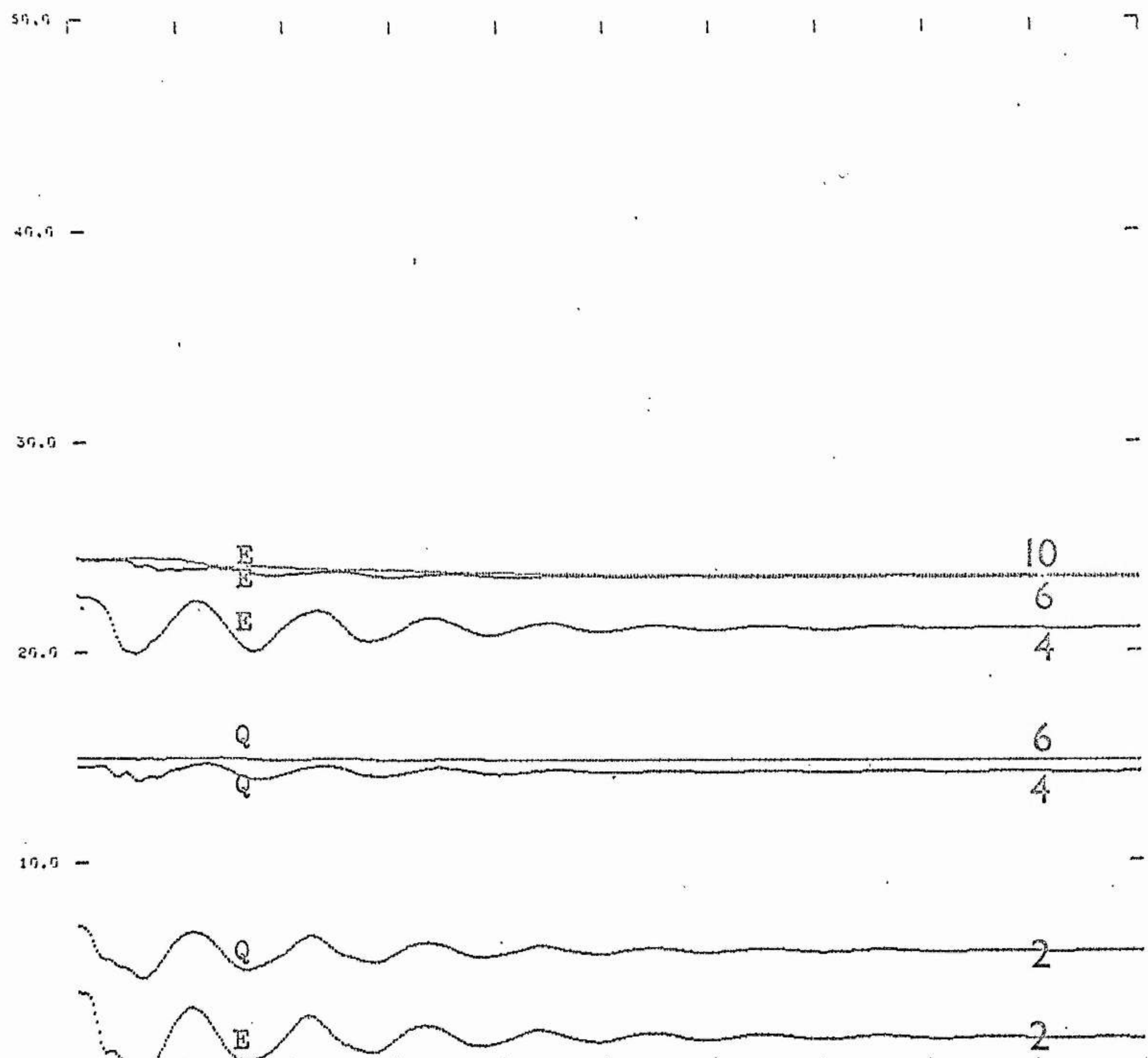
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Fig 39

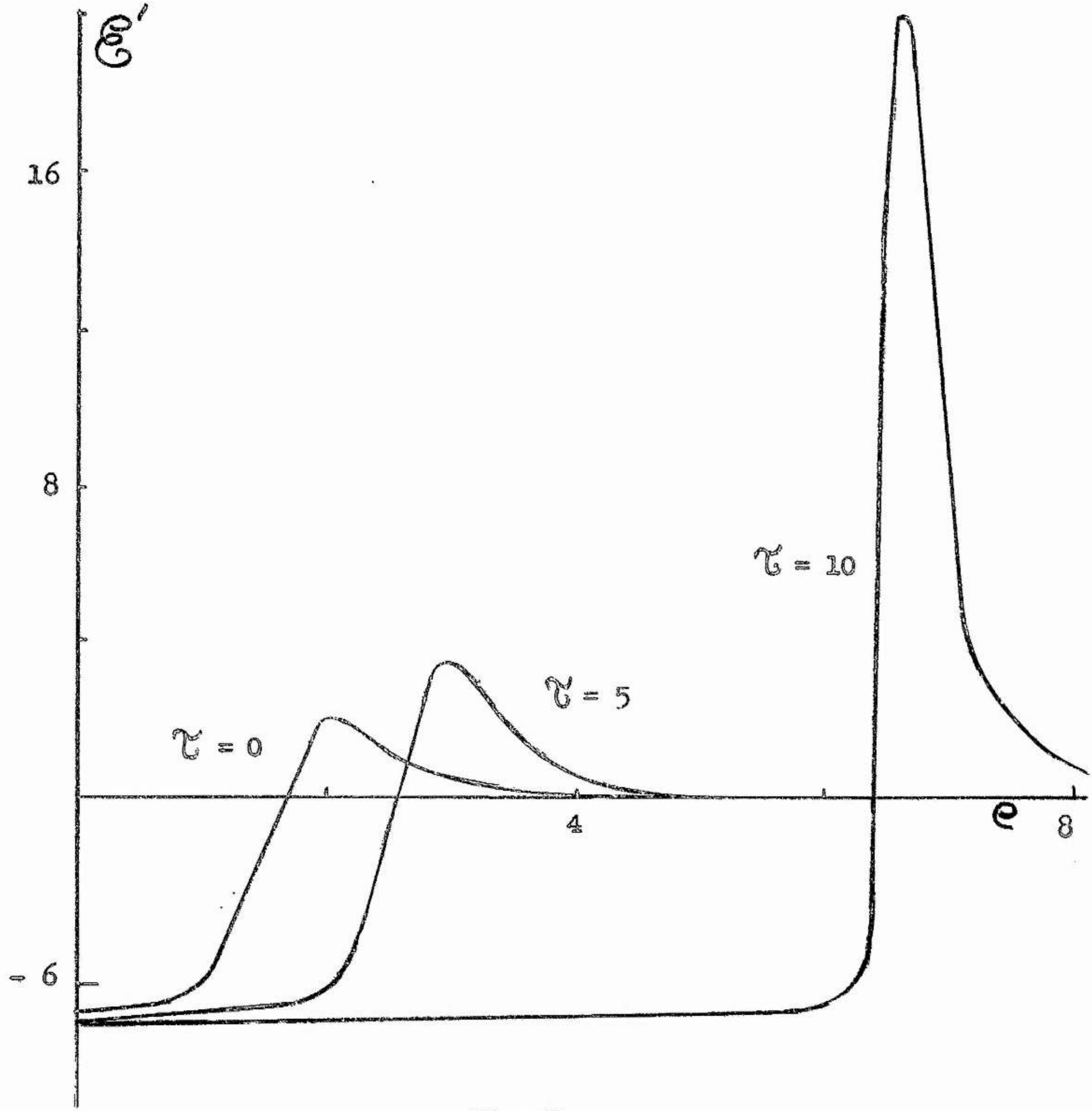


Fig 40

Graph of the reduced energy density against reduced radial distance at a series of reduced times for the point U. $\mathcal{E}'$ tends to assume the value -7.2 over an increasing radius.

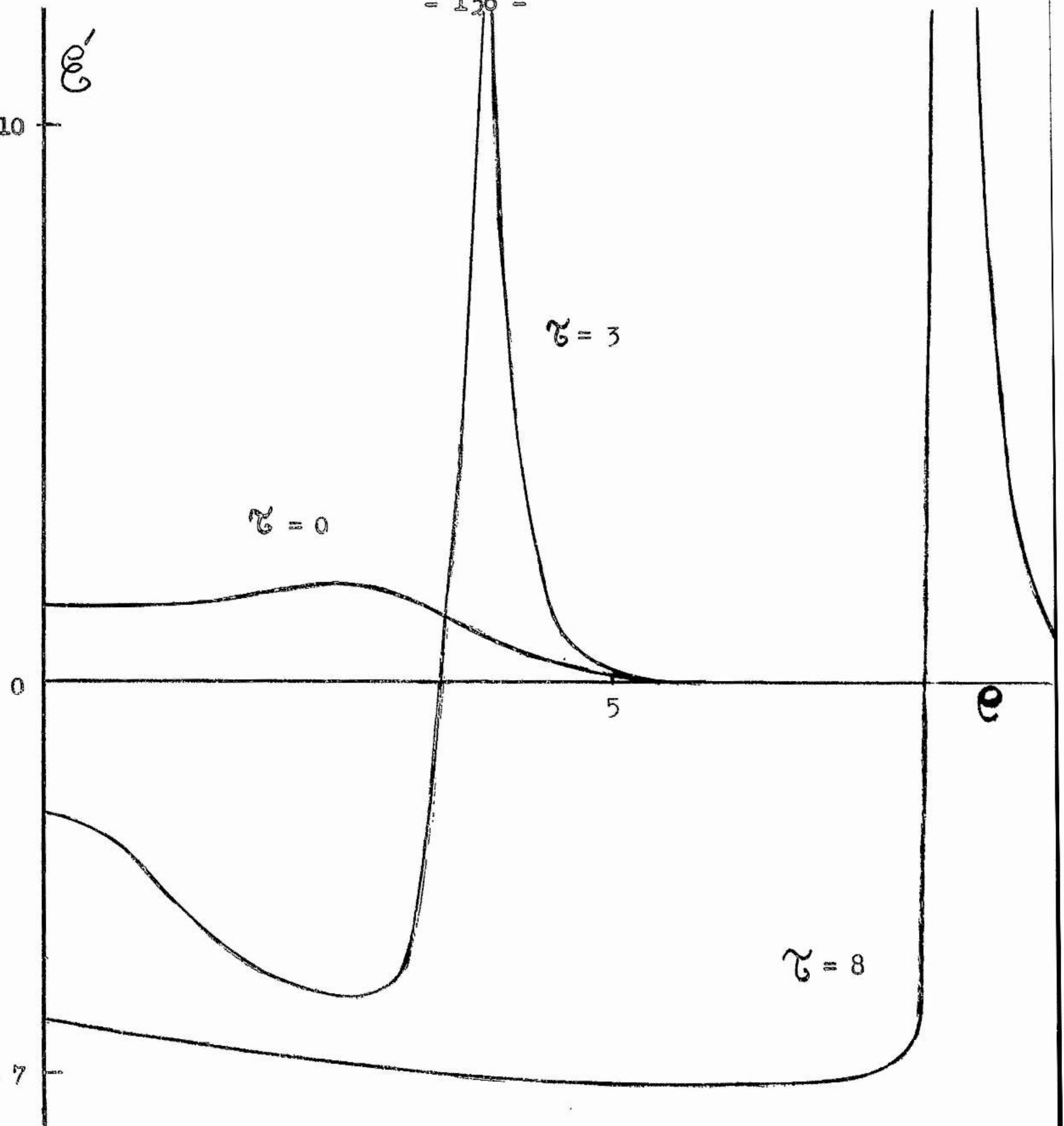


Fig 41

Plot of $\mathcal{E}'$ v ρ for various reduced times for the case $\tilde{B} = 0.1$ Note how the reduced energy density wants to adopt a uniform negative value ~ -7.2

9. DISCUSSION OF THE RESULTS OF CHAPTERS 7 AND 8 AND SIGNIFICANCE OF THE ENERGY.

9.1 Introduction

In chapters 7 and 8 the stability of particle-like solutions for B $>$ 0 was examined by first-order and direct perturbation methods. The agreement between the two methods was found to be good. In particular, the solutions predicted stable by first-order perturbation theory were indeed found to be extremely stable. In chapter 8 we also discounted some simple criteria for stability. In this chapter we suggest a criterion for stability showing that the critical value of $\tilde{\omega}'_c$ is in fact the value of ω' for which E as a function of ω' for a fixed $\tilde{B}$ has a local maximum. E also has a local minimum but this has no obvious significance.

9.2 Some integral relationships

Equation (6.5) can be consistently derived from the variational principle $\delta L = 0$ where L is defined by

$$L = \frac{K}{\mu^2}(1-\omega'^2)^{\frac{1}{2}} \int \left(-\varphi'^2 - \left(\frac{d\varphi'}{dr'}\right)^2 + \frac{1}{2}\varphi'^4 - \frac{B}{3}\varphi'^6\right) d^3r' \tag{9.1}$$

[Equation (9.1) can be obtained by inserting the form $\psi = \varphi e^{-i\omega t}$ into (6.2) and applying the transformation (1.5).]. If we insert $\varphi' = \lambda f(\alpha r')$ into (9.1), where $f(r')$ is defined to be an exact solution of (9.1), then we obtain

$$L = \text{Const} \int \left[-\frac{\lambda^2}{\alpha^3} f^2 - \frac{\lambda^2}{\alpha} \frac{\partial f}{\partial(\alpha r')} + \frac{1}{2}\frac{\lambda^4}{\alpha^3} f^4 - \frac{B}{3}\frac{\lambda^6}{\alpha^3} f^6 \right] (\alpha r')^2 d(\alpha r') \tag{9.2}$$

Then $\left(\frac{\partial L}{\partial \lambda}\right)_{\substack{\lambda=1\\ \alpha=1}} = 0$ implies

$$\int\left(\frac{d\varphi'}{dr'}\right)^2 r'^2 dr' = -\int \varphi'^2 r'^2 dr' + \int \varphi'^4 r'^2 dr' - B\int \varphi'^6 r'^2 dr' \qquad (9.3)$$

Similarly $\left(\frac{\partial L}{\partial \alpha}\right)_{\substack{\lambda=1\\ \alpha=1}} = 0$ implies+

$$\int\left(\frac{d\varphi'}{dr'}\right)^2 r'^2 dr' = -3\int \varphi'^2 r'^2 dr' + \tfrac{3}{2}\int \varphi'^4 r'^2 dr' - B\int \varphi'^6 r'^2 dr' \qquad (9.4)$$

Thus

$$\int \varphi'^4 r'^2 dr' = 4\int \varphi'^2 r'^2 dr' \qquad (9.5)$$

Using (9.3) and (9.4) it is possible to reduce the expression for the energy $E = \int \mathcal{E}\, dV$ where $\mathcal{E}$ is given by (6.6) to

$$E = \frac{4\pi K}{\mu^2}\left[\frac{2\int \varphi'^2 r'^2 dr'}{(1-\omega'^2)^{\frac{1}{2}}} - \frac{2B}{3}(1-\omega'^2)^{\frac{1}{2}}\int \varphi'^6 r'^2 dr'\right] \qquad (9.6)$$

Let us define $E2(B)$, $E6(B)$ by

$$E2(B) = \int \varphi'^2 r'^2 dr'$$

$$E6(B) = \int \varphi'^6 r'^2 dr' \qquad (9.7)$$

+ A similar procedure to this has been used by Rosen[11] in deriving the Pseudovirial Theorem.

Where we have explicitly shown the dependence of E2, E6 on B. Defining by $\bar{E}$ the expression $E\mu^2/4\pi K$ we can express $\bar{E}$ as

$$\bar{E} = \frac{2\,E2(B)}{(1-\omega'^2)^{\frac{1}{2}}} - \frac{2B(1-\omega'^2)^{\frac{1}{2}}\,E6(B)}{3} \qquad (9.8)$$

9.3 Plotting $\bar{E}$ as a function of ω' for fixed $\tilde{B}$

In table 7 the values of E2(B), E6(B) are given for a series of values of positive B. Using these values it is possible to plot the curves of $\bar{E}$ v ω' for a constant $\tilde{B}$. There are two different types of curve.

(i) $\tilde{B} < 3/16$

(ii) $\tilde{B} > 3/16$.

In fig 42 we plot $\bar{E}$ v ω' for various $\tilde{B}$ less than $3/16$. Note that this type of curve is characterised by having a local finite maximum, two local minima, one at $\omega' = 0$, and one at a non zero ω' and an infinite maximum at $\omega' = 1$. The energy curve exists for all ω' in the legitimate range i.e. $0 < \omega' < 1$.

In fig 43 we plot $\bar{E}$ v ω' for various $\tilde{B}$ greater than $3/16$. A typical curve in this case has a local minimum, and two maxima, one at $\omega' = 1$, the other at some lower value of ω' denoted ω'_L which is that value of ω' for which

$$B = \tilde{B}\,(1 - \omega'^2) = 3/16.$$

i.e. $\omega'_L = \sqrt{\dfrac{\tilde{B} - 3/16}{\tilde{B}}}$

For $\omega' < \omega'_L$ there are no particle-like solutions (since this corresponds to a value of B $> 3/16$) and consequently there is no value for $\bar{E}$.

9.4 Significance of the local maximum for $\tilde{B} < 3/16$.

The value of ω', denoted ω'_M, at which $\bar{E}$ has its local maximum for a given $\tilde{B}$, is found to be equal to that value of ω' at which the particle-like solution for the chosen value of $\tilde{B}$ goes stable i.e. to be equal to $\tilde{\omega}'_c$. In table 8 we record the values of ω'_M, $\tilde{\omega}'_c$ for various values of $\tilde{B} < 3/16$. Comparison shows that these values are equal to within numerical error. For B $\to 3/16$, it is difficult to obtain particle-like solutions and in consequence this agreement cannot be tested in this limit ($\omega' \to 0$). Also for $\omega' \to 1$ it is again difficult to obtain accurate values for comparison. However, it seems a reasonable hypothesis to suggest that in fact $\tilde{\omega}'_c$ and ω'_M are truly identical.

It is interesting to note that energy considerations lead one to split up the ($\tilde{B}$, ω') plane into exactly the same regions as stability considerations do. One splits the plane into the regions

i.e. (i) Region of no particle-like solutions

(ii) Region where $\bar{E}$ has two infinite maxima and one local minimum ($\tilde{B} > 3/16$)

(iii) (iv) Region where $\bar{E}$ has two local minima, a local finite maximum and an infinite maximum. The regions (iii) and (iv) are separated/

separated by the line ω'_M Region (iii) corresponds to $\omega' > \omega'_M$ and region (iv) to $\omega' < \omega'_M$

9.5 Apparent lack of significance of the local minimum

Section (9.4) dealt with the significance of the local maximum which occurs for any positive $\tilde{B} < 3/16$. We now consider the local minimum. In this case there is no requirement for $\tilde{B}$ to be less than $3/16$. It is convenient to concentrate the discussion on the value $\tilde{B} = 5/18$, since this is the value of $\tilde{B}$ for the point S considered in section 8.3. A graph of $\bar{\bar{E}}$ v ω' for $\tilde{B} = 5/18$ is shown in fig 43b. Notice the sharpness of the minimum and how, for the point **S**, the energy is considerably above the minimum value.

In general when we disturbed Ψ_{0S} we changed the energy of this state to some new value, $\tilde{E}'$ say. It was found that some of this energy was radiated before the excited state settled down. In fig 44 we plot the values of $\tilde{E}$ to which $\bar{\bar{E}}_S$ is perturbed and also the energy levels to which the disturbed state returns for a number of typical disturbances. Notice that for four decays, E returns to $\bar{\bar{E}}_S$, but in general this does not occur. More important, notice that there is no tendency for a disturbed level to radiate sufficient energy to allow it to decay to the state of minimum energy. It would have been attractive if this <u>had</u> occurred for then the state of lowest energy for a given $\tilde{B}$ would have been the state of maximum stability. We would then have been justified in taking the value of ω' for which this occurred/

occurred as the only significant one, since a particle would never be found in an energy level above this value, being inclined to decay to this lowest level. But alas this does not occur. In a sense the particle-like states are so stable that even when disturbed they do not want to decay even to a lowest level.

9.6 Assignment of Parameters

It was initially hoped that the field equation (6.3) would give rise to particle-like solutions whose lifetime/size ratio would be a function of λ. For the case $\omega' = 0$ it has been shown that this is essentially not so, and that this ratio is approximately a constant independent of λ.

However, for the case $\omega' \neq 0$, our hopes are fulfilled. It has been shown that for certain ω', the particle is stable i.e. the lifetime is infinite, and consequently the lifetime/size ratio is infinite. At the other extreme, in the unstable regions, the lifetime/size ratio is as small as $\sim {}^1/c$

Despite the apparent fulfillment of our hopes, certain defects remain even for this generalised field.

(1) The parameters still have too much freedom i.e. no sensible way of fixing ω' or $\tilde{B}$ emerges from the theory. It would have been encouraging if, for a given $\tilde{B}$, the state of minimum energy had been the state of greatest stability. This would then at least have allowed us to fix ω' for a given $\tilde{B}$. But this does not occur: ω' cannot be fixed/

fixed in this manner.

(2) This field represents a spinless particle. i.e. a boson. In nature there are no stable bosons: all stable particles are fermions. Thus this field cannot sensibly represent a physical particle. One could possibly introduce other fields into the Lagrangian to induce decay, though such ad hoc complications would go against the spirit of the present investigation.

9.7 Summary

Some integral relations satisfied by particle-like solutions are given which allow one to simplify the expression for the reduced energy $\bar{\bar{E}}$. Graphs of $\bar{\bar{E}} \vee \omega'$ for a given $\tilde{B}$ are then drawn. It is found that from such considerations the $(\tilde{B}, \omega')$ plane can be split into the same regions as those resulting from analysing the results of first-order perturbation theory.

For $\tilde{B} > 3/16$, $\bar{\bar{E}}$ has two maxima and a local minimum.

For $\tilde{B} < 3/16$, $\bar{\bar{E}}$ has two local minima separated by a local maximum.

One finds in this latter case, for the range of values of ω' investigated, that there is a direct correlation between the critical values $\tilde{\omega}'_c$ and maximum values ω'_M.

The local minimum which occurs for non zero ω' is considered. It is shown with the aid of results from section 8.3 that this minimum is not as significant as one would have hoped. A suggested method of fixing ω' using this minimum consequently fails.

This/

This field is shown to have certain defects. For example, although we have found solutions which are stable, solutions for which the energy density is positive definite, and solutions for which the lifetime/size ratio is a function of the equation parameters, we still have not found a way of satisfactorily fixing these field parameters - there exists a continuous spectrum of particle masses. Further this field is **unlikely** to be of use in representing elementary particles. The results are in contradiction to natural observations in at least one way. This theory predicts stable spinless massive particles whereas in reality all stable massive particles are Fermions.

The main significance of this work is rather that stable time-dependent particle-like solutions are shown to exist in three space dimensions.

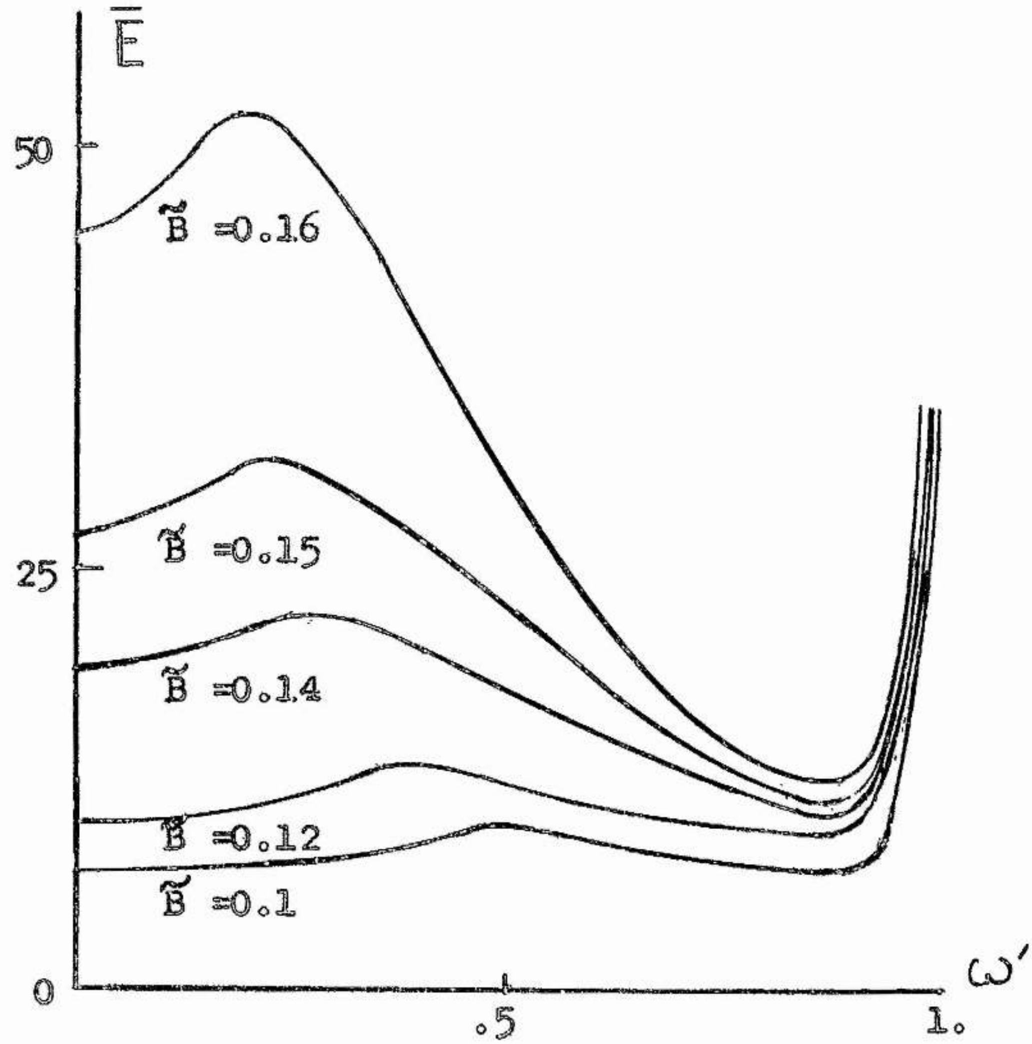


Fig 42

Graph of $\bar{E}$ v ω' for some values of $\tilde{B} < 3/16$
Note that $\bar{E}$ has a local maximum as well as an infinite maximum at $\omega' = 1$.

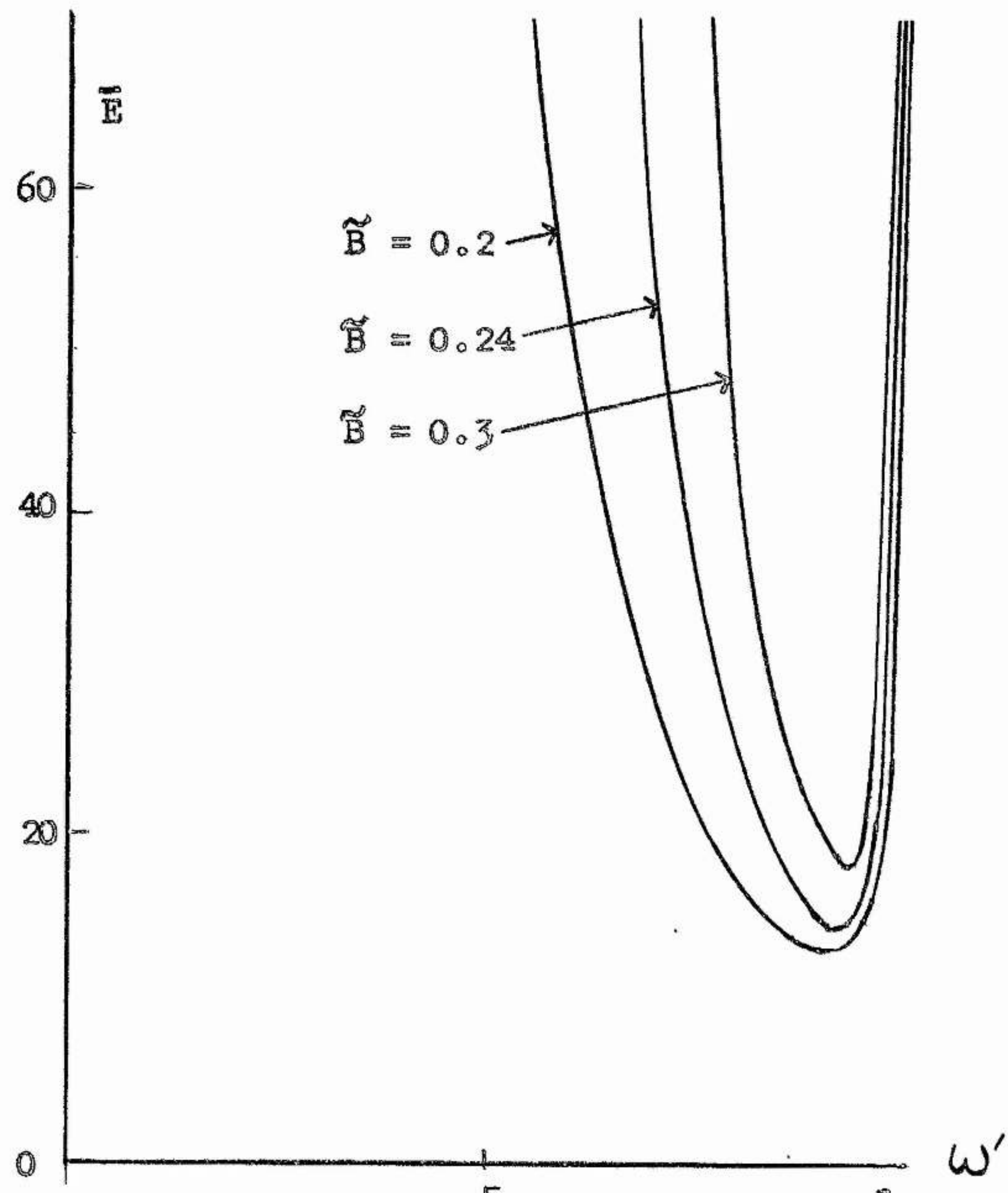


Fig 43

Graphs of $\bar{E}$ v ω' for values of $\tilde{B} > 3/16$, showing how $\bar{E}$ has a sharp local minimum.

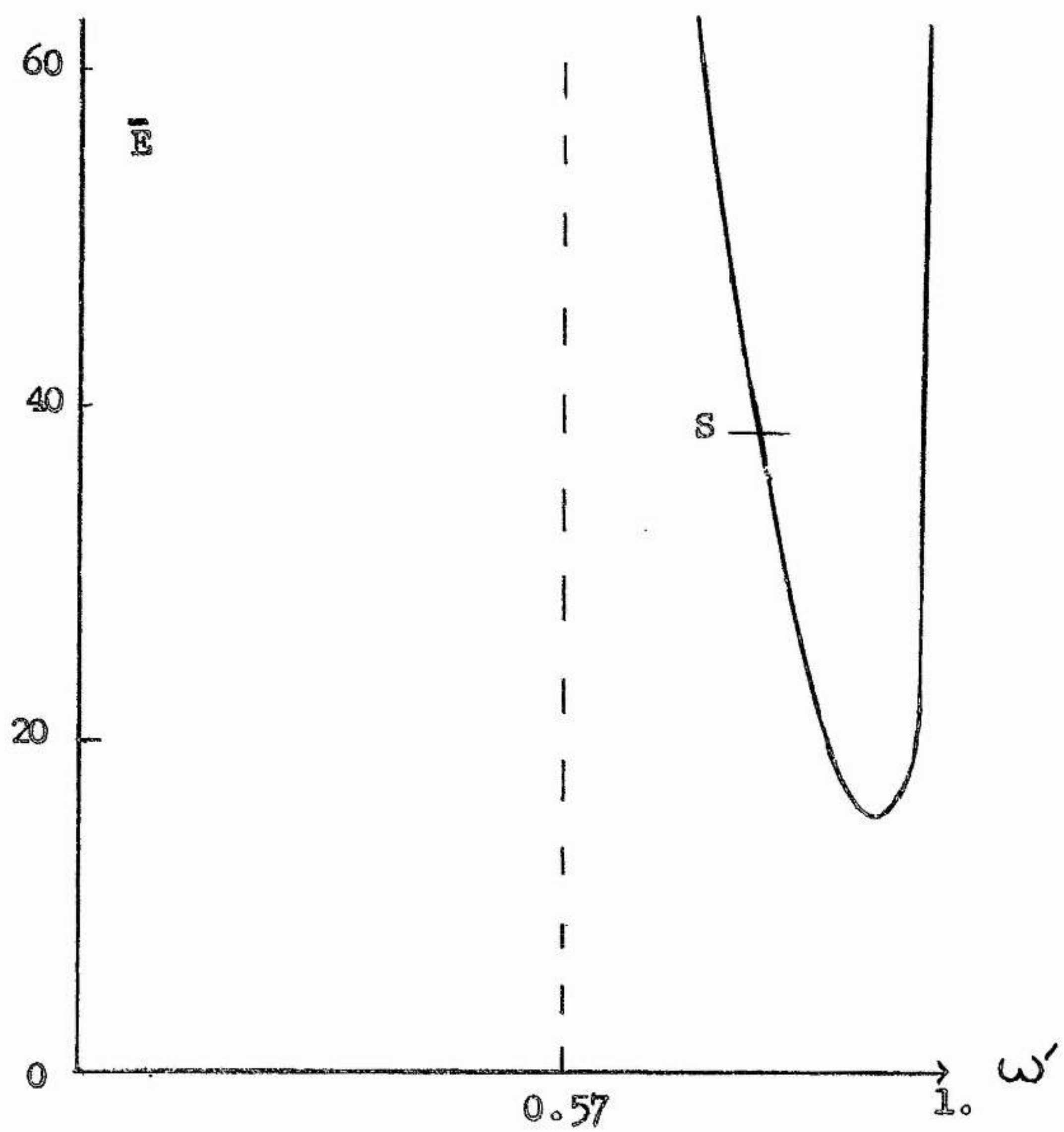


Fig 43b

Plot of $\bar{E}$ v ω' for the case $\tilde{B}$ =5/18. This curve has a minimum at $\omega' = 0.94$ and also has two maxima, one at ω' =1., the other at ω'_L =0.57 For $\omega' < \omega'_L$ there are no particle-like solutions. The point S, ($\omega' = 0.8$) whose stability is considered in section 8.3 is marked. The energy for this point is well above the permitted minimum.

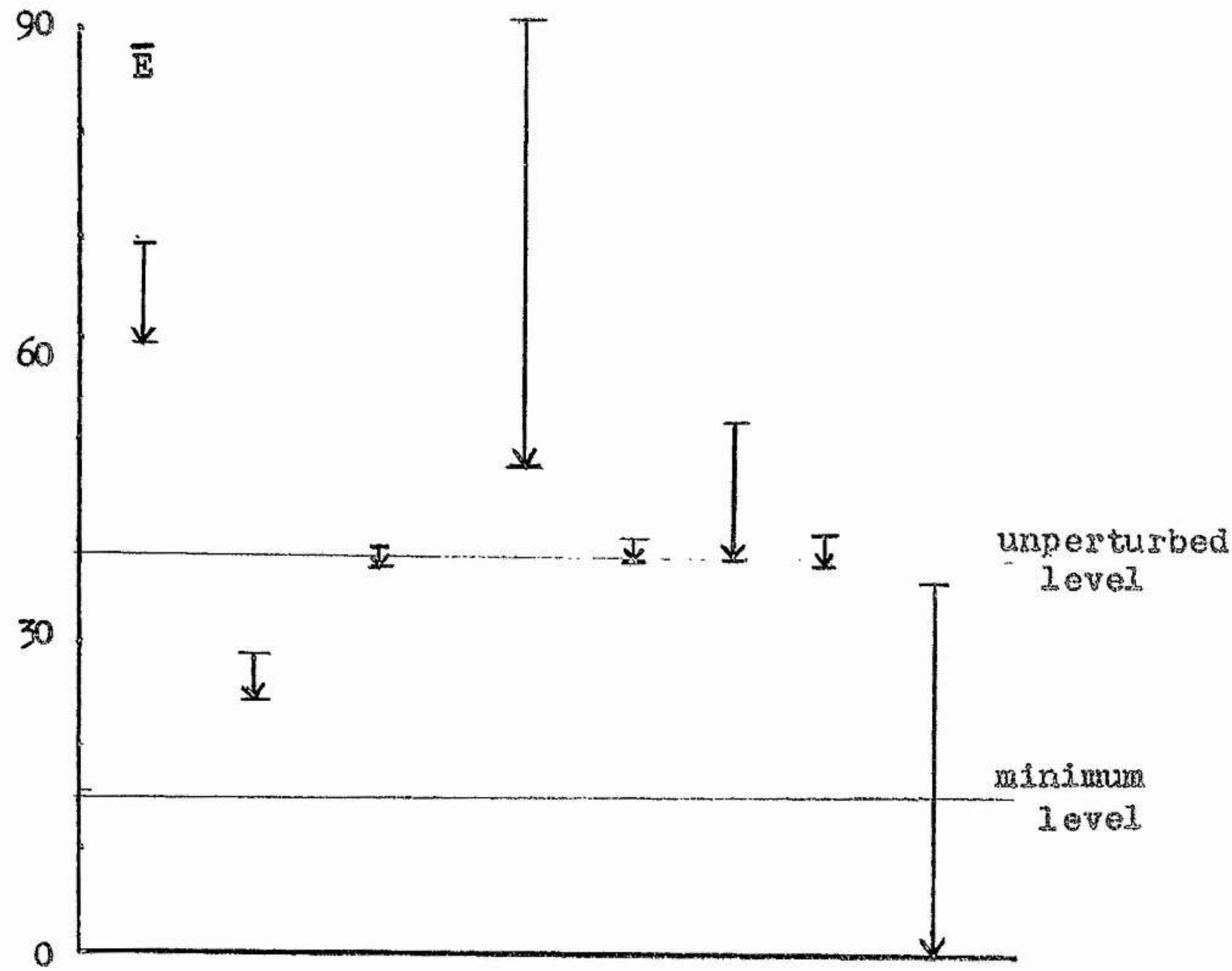


Fig 44

The initial perturbed and final reduced energies for some of the perturbations to S ($\tilde{B}$= 5/18, ω'= 0.8) described in section 8.3 Notice that there is no apparent tendency for a disturbed state to radiate sufficient energy to allow it to drop down to the state of minimum energy for $\tilde{B}$ = 5/18 .

Table 7

Table of values of the integrals $E2 = \int \varphi'^2 r'^2 dr'$, $E6 = \int \varphi'^6 r'^2 dr'$ for the nodeless solution of (6.5) for a series of values of B.

B	E2	E6
0.00	1.50	52.5
0.01	1.80	60.0
0.02	2.17	69.1
0.03	2.63	80.3
0.04	3.23	94.4
0.05	4.02	112.4
0.06	5.07	135.8
0.07	6.49	166.8
0.08	8.49	209.0
0.09	11.4	268.0
0.10	15.7	355.0
0.11	22.4	488.0
0.12	33.6	702.0
0.13	53.6	1077.0
0.14	93.7	1808.0
0.15	187.0	3464.0
0.16	463.0	8300.0

Table 8

$\tilde{B}$	$\tilde{\omega}'_c$	ω'_M
0.16	0.18 ± 0.02	0.19 ± 0.02
0.15	0.19 ± 0.02	0.20 ± 0.02
0.12	0.39 ± 0.01	0.39 ± 0.01
0.1	0.50 ± 0.01	0.51 ± 0.02
0.08	0.71 ± 0.02	0.70 ± 0.03

Table of values of $\tilde{\omega}'_c$ and ω'_M for various values of $\tilde{B}$. To within numerical error these values are equal.

10. SOME SPECULATIONS AND IDEAS FOR FURTHER WORK

10.1 Introduction

In this chapter some suggestions are put forward for continuing the theory. Section 10.2 deals with rotating solutions. This is a crude attempt at approximating non-spherically symmetric particle-like solutions. In section 10.3 we consider a nonlinear non-Lorentz invariant field equation and find that it has particle-like solutions with a whole range of values for the lifetime/size ratio. One defect of the theory so far has been the indeterminate nature of the parameters, particularly ω. In section 10.4 a possible method of fixing ω is suggested by considering multicomponent fields. In section 10.5 we deal with field equations derived from higher-order Lorentz-invariant Lagrangians. For the example chosen it might be possible to obtain stable time-independent particle-like solutions since Derrick's Theorem[6] does not preclude such an occurence.

10.2 Rotating Solutions

So far we have concentrated on spherically symmetric particle-like solutions. Here we consider a variational-numerical method for approximating non-spherically symmetric particle-like solutions to the field equation

$$\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = F(\psi\psi^*)\,\psi \qquad (10.1)$$

An obvious generalisation of (1.3) is a solution of the form

$$\psi = \varphi(r,\theta)\, e^{im\phi}\, e^{-i\omega t}$$

where m is an integer and (r, θ, ϕ) are the spherical polar coordinates. Then (10.1) can be expressed as

$$\frac{d^2\varphi}{dr^2} + \frac{2}{r}\frac{d\varphi}{dr} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\varphi}{\partial\theta}\right) - \frac{m^2 \varphi}{r^2 \sin^2\theta}$$

$$+ \frac{\omega^2}{c^2}\varphi - F(\varphi^2)\varphi = 0 \tag{10.2}$$

If this equation were linear, i.e. $F(\varphi^2)$ were equal to a constant, then a partition of $\varphi(r, \theta)$ into $R(r)\ \Theta(\theta)$ would separate (10.2) and $\Theta(\theta)$ could then be expressed in terms of the usual Legendre polynomials. When (10.2) is not linear then this separation does not work. However (10.2) can be obtained from the variational principle $\delta I = 0$ where I is given by

$$I = \int_0^\infty r^2 dr \int_0^\pi \sin\theta\, d\theta \left\{ \frac{1}{2}\left(\frac{d\varphi}{dr}\right)^2 + \frac{1}{2r^2}\left(\frac{\partial\varphi}{d\theta}\right)^2 \right.$$

$$\left. - \frac{1}{2}\left(\frac{\omega^2}{c^2} - \frac{m^2}{r^2\sin^2\theta}\right)\varphi^2 + \frac{1}{2} f(\varphi^2) \right\}$$

$$F(\varphi^2) = \frac{d f(\varphi^2)}{d(\varphi^2)} \tag{10.3}$$

Let us consider the case

F/

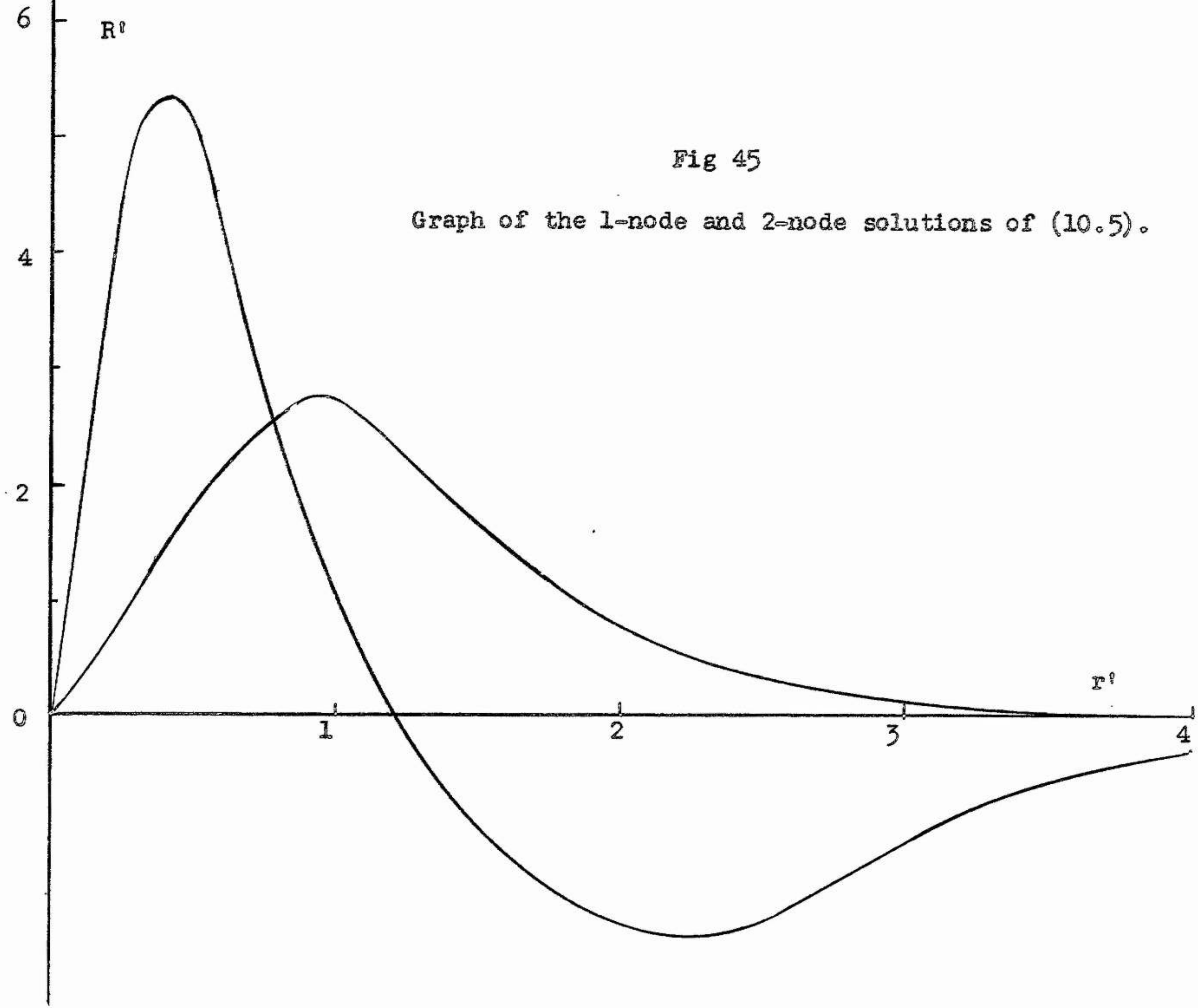


Fig 45

Graph of the 1-node and 2-node solutions of (10.5).

$$F(\varphi^2) = K^2 - \mu^2\varphi^2 \qquad (10.4)$$

If $F(\varphi^2)$ were a constant then solutions would be of the form

$$j_\ell(r) \; P_{\ell m}(\cos\theta)$$

Let us try a variational solution for the case (10.4) of the form

$$\varphi = R(r)\,\mathrm{Sin}\,\theta \qquad \text{(taking m = 1)}$$

Then the θ dependence of (10.3) can be integrated out, and after transforming the resultant Euler-Lagrange equation in $R(r)$ the problem can be reduced to solving a second order differential equation of the form

$$\frac{d^2R'}{dr'^2} + \frac{2}{r'}\frac{dR'}{dr'} - \frac{2}{r'^2}R' = R' - R'^3 \qquad (10.5)$$

which can be seen to be a generalisation of (1.6). In this case we require $R' = 0$ at $r' = 0$, and $R' \rightarrow 0$ as $r' \rightarrow \infty$. Equation (10.5) was solved numerically. There is probably an infinity of solutions with 1, 2, 3, . . . nodes. (There cannot be a nodeless solution). The two solutions of lowest order are shown in fig 45.

10.3 A Nonlinear Schrödinger Equation

Although Lorentz-invariance seems a sensible postulate, we disregard it in this section and consider a nonlinear Schrödinger equation. The equation considered is

$$\nabla^2\psi + \frac{2mi}{\hbar}\frac{\partial\psi}{\partial t} = K^2\psi - \mu^2\psi\psi^*\psi \qquad (10.6)$$

If we consider particle-like solutions of the form

$$\psi = \varphi(r)\, e^{-i\omega t} \tag{10.7}$$

we can reduce the problem of finding $\varphi(r)$ to solving

$$\frac{d^2\varphi'}{dr'^2} + \frac{2}{r'}\frac{d\varphi'}{dr'} = \varphi' - \varphi'^3 \tag{10.8}$$

where

$$\varphi' = \mu\,\varphi\,\left(K^2 - \frac{2m\omega}{\hbar}\right)^{-\frac{1}{2}}$$

$$r' = \left(K^2 - \frac{2m\omega}{\hbar}\right)^{\frac{1}{2}} r \tag{10.9}$$

Now (10.8) is exactly (1.6) whose solutions we found in chapter 2. If we examine the stability of the nodeless solution to (10.6) of the form (10.7) we find that it is unstable, and that the lifetime/size ratio is proportional to $(1 - \omega')^{-\frac{1}{2}}$ where $\omega' = \frac{2m\,\omega}{\hbar K^2}$. Since the parameter ω' can range anywhere from +1 to $-\infty$ we have a whole continuous range of values for the lifetime/size ratio.+

10.4 Multicomponent Solutions

In chapters 1 - 9 we considered particle-like solutions of the form

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = F(\psi, \psi^*)\,\psi \tag{10.10}$$

+ When $K^2=0$, the lifetime/size ratio is proportional to $(-\bar{\omega}')^{-\frac{1}{2}}$ where $\bar{\omega}' = 2m\omega/\hbar$, and lies in the range $0 > \bar{\omega}' > -\infty$.

where ψ was a 1-component field. For the cases considered, it was found that the field parameters were too indeterminate. No method of sensibly fixing the parameters was found. It had been initially hoped that for solutions of the form

$$\psi = \varphi(r)\, e^{-i\omega t}$$

stability considerations would allow us to determine ω. It may be, that for some particular $F(\psi, \psi^*)$, this will occur. However, an alternative method is suggested here whereby ω' would be determined not by stability methods but perhaps by considerations of the very existence of particle-like solutions. [This could also conceivably occur for (10.10)] We can generalise the form (10.10) by considering, for example,

$$\nabla^2 \underset{\sim}{\psi} - \frac{1}{c^2} \frac{\partial^2 \underset{\sim}{\psi}}{\partial t^2} = K^2 \underset{\sim}{\psi} - \mu^2 \underset{\sim}{\psi} \underset{\sim}{\psi}^{\dagger} \underset{\sim}{\psi} \qquad (10.11)$$

where $\underset{\sim}{\psi}$ is now a multicomponent vector. The problem arises as to what one should take for ψ and what the components of ψ represent. For illustration, let us consider

$$\underset{\sim}{\psi} = \begin{pmatrix} \varphi_1(r)\, e^{-i\omega t} \\ \varphi_2(r) \\ \varphi_1(r)\, e^{+i\omega t} \end{pmatrix} \qquad (10.12)$$

where the three components of ψ might conceivably be components of isospin. Inserting (10.12) into (10.11) and making transformations on φ and r, one can reduce (10.11) to a coupled set of nonlinear equations of the form

$$\nabla'^2 \varphi'_1 = \varphi'_1 - (2\varphi_1'^2 + \varphi_2'^2)\,\varphi'_1$$

$$\nabla'^2 \varphi'_2 = \alpha^2 \varphi'_2 - (2\varphi_1'^2 + \varphi_2'^2)\,\varphi'_2 \qquad (10.13)$$

where $\alpha^2 = 1/(1 - \omega'^2)$ with $\omega' = \omega/KC$.

It is possible that α^2 might act as some sort of eigenvalue i.e. that solutions to (10.13) might exist only for certain discrete values of α. In such an event this would immediately fix ω. However no attempt has been made to solve (10.13) so that it remains an open question whether α is indeed restricted to discrete values.

10.5 Higher-Order Lorentz-invariant Lagrangians

It was suggested by Derrick in 1964 that it might be possible to get stable time-independent particle-like solutions in three space dimensions if we choose higher-order Lagrangians. Let us consider a Lagrangian of the form

$$L = \int \left\{ \left[|\nabla\psi|^2 - \frac{1}{c^2}\left|\frac{\partial\psi}{\partial t}\right|^2 \right]^n + f(|\psi|^2) \right\} d^3r$$

For the case n = 2 the Euler-Lagrange equation derived from the above Lagrangian with

$$f(\psi) = Z^{-8}\left(2Z^4\psi^8 - \frac{26gZ^2\psi^{10}}{5} + 3g^2\psi^{12}\right)$$

has/

has a real time-independent particle-like solution

$$\psi = \frac{Z}{(g + r^2)^{1/2}}$$

This is essentially the same form of solution as Rosen[7] obtained for the field he examined but in this case there is a possibility that ψ will be stable, whereas in Rosen's field ψ was necessarily unstable. The stability of particle-like solutions to higher-order Lagrangians requires to be examined.

11. SUMMARY

In this thesis we are primarily concerned with the stability of time-dependent particle-like solutions to some nonlinear field equations. The particular form of time dependence examined is

$$\psi = \varphi(r)\, e^{-i\omega t} \tag{1.3}$$

In chapters 1 and 2 the existence of particle-like solutions of this form is investigated for the field

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = K^2\psi - \mu^2\psi\psi^*\psi \tag{1.1}$$

where the problem is reduced to finding solutions of the ordinary nonlinear second-order differential equation

$$\frac{d^2\varphi'}{dr'^2} + \frac{2}{r'}\frac{d\varphi'}{dr'} = \varphi' - \varphi'^3 \tag{1.6}$$

This equation has an infinity of solutions having respectively, 0, 1, 2, . . . nodes. The first eight solutions are obtained numerically.

A first-order perturbation method for examining the stability of solutions of the form (1.3) is proposed in chapter 3. The problem of stability is reduced to solving a complex eigenvalue problem (3.3). If, for a given ω', there exists no eigenvalue (Ω') with a nonzero imaginary part, then the particle-like solution for that ω' would be stable. In fact all eigenvalues Ω' of (3.3) for the nodeless particle-like solution of (1.6) are found to be purely imaginary irrespective of the value of ω', implying that all such "particles" are unstable. Variational calculations are also made in chapter 3 which give results in agreement with those obtained numerically from first-order perturbation/

perturbation theory.

The stability of the lowest-order solution is then examined in chapter 4 by direct perturbation methods. Again we find that the solution is unstable for both zero and nonzero ω. This direct method further indicates that the solution can be unstable in two distinct ways, a result not obtainable from first-order perturbation theory. The "particle" can decay by radiating all its energy (the dissipative decay mode) or by drawing an increasing amount of negative energy into a 'hole' around its centre (the singular decay). This latter decay is a consequence of the field allowing the energy density to be negative in certain regions of space. The dissipative decay mode is considered physically acceptable, the singular mode unacceptable.

Apart from this unsatisfactory mode of decay there are other deficiencies in this field. The lifetime/size ratio is a constant independent of the field parameters K, μ, ω with a value far removed from that of the metastable particles though possibly of the correct size for the highly unstable particles (lifetimes $\sim 10^{-23}$ secs). Further, no satisfactory method of fixing these field parameters has been found, though some tentative suggestions are made in chapter 5.

In an attempt to overcome such deficiencies a generalisation of the initial field is considered in chapters 6 - 9:

$$\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = K^2 \psi - \mu^2 \psi \psi^* \psi + \lambda \psi \psi^* \psi \psi^* \psi$$

The problem of finding particle-like solutions of the type $\varphi\, e^{-i\omega t}$ to this generalised field is reduced to solving the equation

$$\frac{d^2\varphi'}{dr'^2} + \frac{2}{r'}\frac{d\varphi'}{dr'} = \varphi' - \varphi'^3 + B\,\varphi'^5$$

where

$$B = \lambda\left(K^2 - \omega^2/c^2\right)/\mu^4$$

It is shown that this equation probably has particle-like solutions of the same form as the B = 0 case (i.e. solutions with 0, 1, 2, . . . nodes) for all values of B in the range $-\infty < B < 3/16$, though our attention is primarily concentrated on the region $0 < B < 3/16$, since we seek solutions for which the energy density is positive definite. (This can only occur if $B > 0$).

The stability of the nodeless solutions is examined by methods analagous to those used for the $\lambda = 0$ case. Stable time-dependent particle-like solutions are shown to exist, some of which have the added attraction of positive definite energy density.

Despite the apparently attractive features of this generalised field, certain defects remain. There is no satisfactory method of fixing field parameters. There are no stable spinless massive particles found in nature. The main significance of this work is not that we have found a field capable of representing an elementary particle, but rather that the existence of stable particle-like solutions in three/

three space dimensions is shown possible. Further refinement is necessary before we have a physically acceptable field theory. In chapter 10 some suggestions for further study are made.

Appendix A

Series solution of (1.6) for small r'

The solution of the equation (1.6) can be approximated for small r' as a power series in r'

$$= a_o + a_2 r'^2 + a_4 r'^4 + a_o r'^6 + \ldots a_{2n} r'^{2n} + \ldots$$

where

$$\begin{aligned}
a_2 &= a_o(1 - a_o^2)/6 \\
a_4 &= a_2(1 - 3a_o^2)/20 \\
a_6 &= (a_4 - 3a_o^2 a_4 - 3a_o a_2^2)/42 \\
a_8 &= (a_6 - 6a_o a_2 a_4 - 3a_o^2 a_6 - a_2^3)/72 \\
a_{10} &= (a_8 - 6a_o a_2 a_6 - 3a_o^2 a_8 - 3a_o a_4^2 - 3a_2^2 a_4)/110 \\
a_{12} &= (a_{10} - 6a_o a_2 a_8 - 6a_o a_4 a_6 - 3a_o^2 a_{10} - 3a_2^2 a_6 - 3a_2 a_4^2)/156 \\
a_{14} &= (a_{12} - 6a_o a_2 a_{10} - 6a_o a_4 a_8 - 6a_2 a_4 a_6 - 3a_o a_6^2 - 3a_o^2 a_{12} \\
&\qquad - 3a_2^2 a_8 - a_4^3)/210 \\
a_{16} &= (a_{14} - 6a_o a_2 a_{12} - 6a_o a_4 a_{10} - 6a_o a_6 a_8 - 6a_2 a_4 a_8 - 3a_o^2 a_{14} \\
&\qquad - 3a_2^2 a_{10} - 3a_2 a_6^2 - 3a_4^2 a_6)/272
\end{aligned}$$

The number of terms required in the series expansion depends on the mesh size, and on the order of the solution being sought. In general, for smaller mesh size one needs fewer terms, but for increasing order solutions more terms.

Appendix B

Method of matching-in-the-middle applied to second-order nonlinear differential equations

Let us consider the equation

$$\frac{d^2 Z}{dr'^2} = Z - Z^3/r'^2 \tag{B1}$$

The boundary conditions for this equation are $Z = 0$ at $r' = 0$ and $Z \rightarrow 0$ as $r' \rightarrow \infty$. Let us denote by Z_f the forward integrated solution, by Z_b the backward integrated solution and by r'_c the value of r' at which we require to 'join' the solutions.

This implies

$$\left(Z_b\right)_{r'_c} = \left(Z_f\right)_{r'_c}$$
$$\left(\frac{dZ_b}{dr'}\right)_{r'_c} = \left(\frac{dZ_f}{dr'}\right)_{r'_c} \tag{B2}$$

A numerically equivalent condition to (B2) is to match Z_f and Z_b at two neighbouring points, r'_c and r'_{c+1}. Then the matching condition is

$$\left(Z_b\right)_{r'_c} = \left(Z_f\right)_{r'_c}$$
$$\left(Z_b\right)_{r'_{c+1}} = \left(Z_f\right)_{r'_{c+1}} \tag{B3}$$

For linear second order differential equations, it is possible to adjust the parameters of the attempted solution by a generalised Newton-type correction process until (B3) is satisfied. For nonlinear equations this/

this method is less useful. With some modification it was possible to obtain solutions to (B1) by such a method, but a much better way of doing so will be described. This method may be denoted the minimisation correction process.

For small r' the series solution of (B1) is

$$Z = a_o r' + a_2 r'^3 + a_4 r'^5 + \dots\dots\dots$$

where a_2, a_4 . . . are given in Appendix A.
For large r' the asymptotic form of (B1) is $Z \rightarrow de^{-r'}$. The problem is to find those values of a_o and d which give matched solutions. Let us construct the function

$$F = (Z_b - Z_f)^2_{r'=r'_c} + (Z_b - Z_f)^2_{r'=r'_{c+1}} \qquad \text{(B4)}$$

Then F is a function of the parameters (a_o, d) which is positive semi-definite and has a minimum value of zero only when the forward and backward solutions are matched. The problem has thus been modified to minimising a function of two variables to zero. A method of doing this is given in Appendix C. It is found that this approach is very successful, and matched solutions to (B1) and (1.6) were readily found. This method was also later used to solve (6.1).

Appendix C

Function Minimisation

The problem of minimising a function F of n variables (x_i, i = 1,n) falls into two categories.

(i) functions for which both the function F, and the first derivatives $\frac{\partial F}{\partial x_i}$ (i = 1,n) can be calculated.

(ii) functions for which F can be calculated, but for which the derivatives cannot.

Satisfactory methods have been given by Fletcher and Powell[13], and by Fletcher and Reeves[14] for the first of these. Methods have also been given for (ii) but many of the early methods are dependent on the nature of the function F. A review of this subject is given by R. Fletcher[15].

For the problem we want to solve by minimisation, it is possible in an artificial way to evaluate derivatives of F, with respect to the parameters of F. However, it would be preferable to have a method which did not require the evaluation of derivatives of F. A recent paper by Nelder and Mead[16] suggests a simplex method for function minimisation which does not require the derivatives of F. This method was programmed and found to be extrememly efficient, better than any other method to date, and is the method adopted for all the numerical minimisation work in this thesis.

Appendix D

Non square integrable solutions

If either γ^2 or δ^2 is -ve in (3.7) then the solutions to (3.7) are not square integrable. However it is possible to construct a square integrable solution by taking linear combinations of the non-square integrable solutions. Consider the analogy:

If one throws a particle from ∞ at a scattering centre, potential $V(r)$, with momentum in a narrow band about $\hbar \underset{\sim}{k}_0$, it will be represented by a wave packet (square integrable) of the form

$$\int d^3\underset{\sim}{k}\, f(\underset{\sim}{k})\, \psi_{\underset{\sim}{k}}(\underset{\sim}{r})\, e^{-\frac{i\hbar k^2 t}{2m}}$$

where $f(\underset{\sim}{k})$ is peaked about $\underset{\sim}{k}_0$ and $\psi_{\underset{\sim}{k}}(\underset{\sim}{r})$ is a positive energy eigenfunction (not square integrable) of

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right]\psi_{\underset{\sim}{k}}(\underset{\sim}{r}) = \frac{\hbar^2 k^2}{2m}\psi_{\underset{\sim}{k}}(\underset{\sim}{r})$$

with asymptotic form $\psi_{\underset{\sim}{k}}(\underset{\sim}{r}) \rightarrow \text{Const}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}$ as $r \rightarrow \infty$ provided $V(r)$ is of finite range.

For $V(r)$ attractive and short range we would expect $-\frac{\hbar^2}{2m}\nabla^2 + V(r)$ to have a finite number of bound states $\psi_i(\underset{\sim}{r})$ with negative energies E_i and a continuum of positive energy non-square integrable eigenfunctions $\psi_{\underset{\sim}{k}}(\underset{\sim}{r})$ satisfying the usual orthogonality and completeness relations. One can then express the general solution of the time-dependent Schrödinger equation as

$$\psi(\underset{\sim}{r},t) = \sum_i f_i\, \psi_i(\underset{\sim}{r})\, e^{-\frac{iE_i t}{\hbar}} + \int d^3\underset{\sim}{k}\, f(\underset{\sim}{k})\, \psi_{\underset{\sim}{k}}(\underset{\sim}{r})\, e^{-\frac{i\hbar k^2 t}{2m}}$$

Thus for such a problem it is necessary to consider the +ve energy eigenfunctions $\psi_{\underset{\sim}{k}}(\underset{\sim}{r})$ even though they are not themselves square integrable.

This suggests that a similar situation might occur for the present problem, equation (3.1). Perhaps there exists a discrete set of Ω_j corresponding to the square integrable solutions of (3.2) together with a continuum of Ω corresponding to non-square integrable solutions. The general solution to (3.1) would then be

$$\sum_j f_j \left(\eta_j(\underset{\sim}{r}) e^{-i\Omega_j t} + \chi_j^*(\underset{\sim}{r}) e^{i\Omega_j^* t} \right) e^{-i\omega t} +$$

$$\int d\Omega \; f_\Omega \left(\eta(\underset{\sim}{r}) e^{-i\Omega t} + \chi^*(\underset{\sim}{r}) e^{i\Omega^* t} \right) e^{-i\omega t}$$

which could still be square integrable. In chapter 3 we concentrated on square integrable solutions of (3.1). If we now seek non-square integrable solutions of (3.1) this can be shown to be equivalent to seeking solutions of (3.5), (3.6) for values of γ^2, δ^2 which are not both positive. We consider two cases

(i) γ^2 -ve, δ^2 -ve
(ii) γ^2 -ve, δ^2 +ve.

In case (i) we do find a continuum of Ω values corresponding to non-square integrable solutions, and for each Ω there exist two linearly independent solutions.

In case (ii) we again find a continuum of Ω values but this time there is only one solution for a given Ω.

Appendix E

Method of solving differential eigenvalue problems using a minimisation routine

Let us consider a simple eigenvalue problem, say (3.9)

$$\left[\frac{d^2}{dr'^2} - \gamma^2 + 3\varphi_o'^2\right] z = 0 \qquad \text{(E1)}$$

This has asymptotic form $z \longrightarrow e^{-\gamma r}$. The problem is to find that value of γ for which $z_{r'=0} = 0$.

This problem can be tackled by using a generalised Newton-type correction process, and in fact such a method works well for this case. However, a Newton-type method does not lend itself to complicated simultaneous differential eigenvalue problems, particularly where the number of parameters to be corrected exceeds the number of boundary conditions. A minimisation method of solving differential eigenvalue problems is explained which can easily be generalised to tackle much more complicated problems.

Let us consider (E1). One would take a trial value of γ, and use this to compute the asymptotic form of z. Then one would numerically integrate (E1) from large r' to r' = 0 and evaluate the function

$$F = (z)^2_{r'=0}$$

Only when F attains its minimum value of zero do we have a solution. Thus, the correction process consists of minimising F as a function of/

of γ until F is zero.

This method is easily adapted to solve more complicated problems such as (3.5) when Ω' is complex. The asymptotic form of (3.5) is in this case a complicated expression involving the five parameters Ω_r', Ω_i', E, P, Q, while there are only four boundary conditions $g_{r,} = g_{i,} = f_{r,} = f_i = 0$ at $r' = 0$. We now minimize $F = [g_r^2 + g_i^2 + f_r^2 + f_i^2]_{r'=0}$ with respect to the five parameters Ω_r', Ω_i', E, P, Q. The fact that the number of free parameters exceedsthe number of boundary conditions does not lead to any difficulty with the minimization method, in contrast to Newton-type correction procedures. Indeed, to allow for the possibility that additional special solutions might exist at certain ω' the programme was also run allowing ω' to vary as well. The method can obviously be generalized to more complicated problems by taking F as the sum of squares of the quantities required to vanish by the boundary conditions.

Appendix F

Solutions of (3.9)

In Section 3.2.1 we gave the results of solving (3.9) numerically using a good numerical approximation for φ_0'. In this appendix as an independent check we consider solving (3.9) using a crude variational approximation for φ_0'. Betts, Schiff, Strickfaden[10] have found both simple and sophisticated variational approximations to the nodeless solution of (1.6). We consider a particularly simple one

$$\varphi_0' = 4\sqrt{2}\, e^{-\sqrt{3}r'} \tag{F1}$$

The lowest eigenvalue ($-\gamma^2$) of (3.9) is given by the variational principle

$$-\gamma^2 \leqslant \frac{\int_0^\infty \left[\left(\frac{dz}{dr'}\right)^2 - 3\varphi_0'^2 z^2\right] dr'}{\int_0^\infty z^2\, dr'} \tag{F2}$$

the equality holding only if z satisfies (3.9).

The simplest form of solution to (3.9) satisfying the boundary conditions $z = 0$ at $r' = 0$, $z \to 0$ as $r' \to \infty$ is a solution with one node which can be crudely represented by a trial wave function $r' e^{-\alpha r'}$. Using this trial wave function and the above variational approximation for φ_0' yields the approximate value for γ^2 of 16.7 leading to a value for Ω_u^0 of 3.96, to be compared with the value of 3.95 obtained by solving (3.9) numerically using a numerical approximation for φ_0'.

When φ_0' is approximated by (F1) it is possible to solve (3.8), (3.9) in terms of Bessel functions. For example the transformation

X/

$X = 4\sqrt{2}\, e^{-\sqrt{3}r'}$ brings (3.9) to the standard form of Bessel's equation and gives a value for γ^2 of 17.4 as well as the spurious value $\gamma^2 = .05$. This latter solution is a consequence of taking too crude an approximation for φ_0' and is not a true solution of (3.9).

Appendix G

No real solutions to (3.6) for $\ell > 1$, $\omega' = 0$

When $\omega' = 0$, (3.6) can be written

$$\left[\frac{d^2}{dr'^2} - \frac{\ell(\ell+1)}{r'^2} - \gamma^2 + 3\varphi_0'^2\right] z = 0 \qquad (G1)$$

$$\left[\frac{d^2}{dr'^2} - \frac{\ell(\ell+1)}{r'^2} - \gamma^2 + \varphi_0'^2\right] y = 0 \qquad (G2)$$

where $z = g_\ell + f_\ell$ $\qquad y = g_\ell - f_\ell$

It is here shown that there are no square integrable solutions to (G1) or (G2) for any $\ell > 1$. Let us consider (G1). The asymptotic form is $z \rightarrow e^{-\gamma r'}$ and the boundary condition at $r' = 0$, is $z = 0$. Thus the simplest form of solution to (G1) is a solution with 1 node, the next simplest a solution with two nodes etc. The two lowest order solutions are shown in fig G1 . Now any solution of (G1) of this form must have a point of inflexion at which z is non zero (denoted by A in fig. G1). At a point of inflexion $\frac{d^2 z}{dr'^2} = 0$ and since $z \neq 0$ this implies

$$- \frac{\ell(\ell+1)}{r'^2} - \gamma^2 + 3\varphi_0'^2 = 0$$

$$- \ell(\ell+1) + 3Z^2 > 0 \qquad (G3)$$

where $Z = r'\varphi_0'$

Plots of φ' v r' and Z v r' for the lowest order (nodeless) solution of (1.6) are shown in fig. G2. From this graph one can see that Z has a maximum value of 1.2, and in consequence that G3 can be satisfied only for $\ell = 0$, 1 but not for $\ell > 1$. If one considers (G2), a similar analysis shows that it can have solutions only for $\ell = 0$.

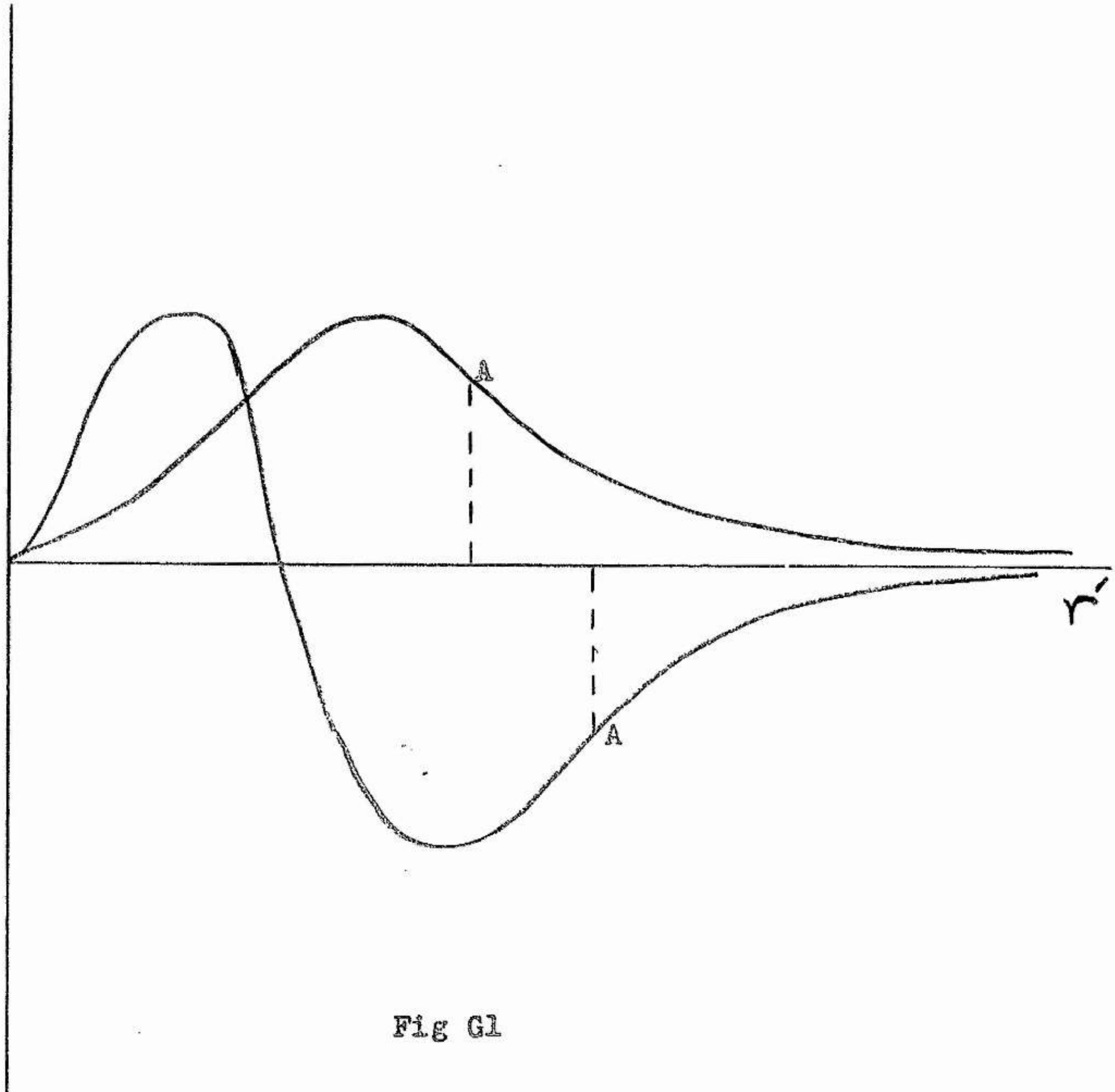


Fig G1

Forms of solutions to (G1) having respectively 1, 2 nodes. Such solutions must have a point of inflexion at which the function is non zero.

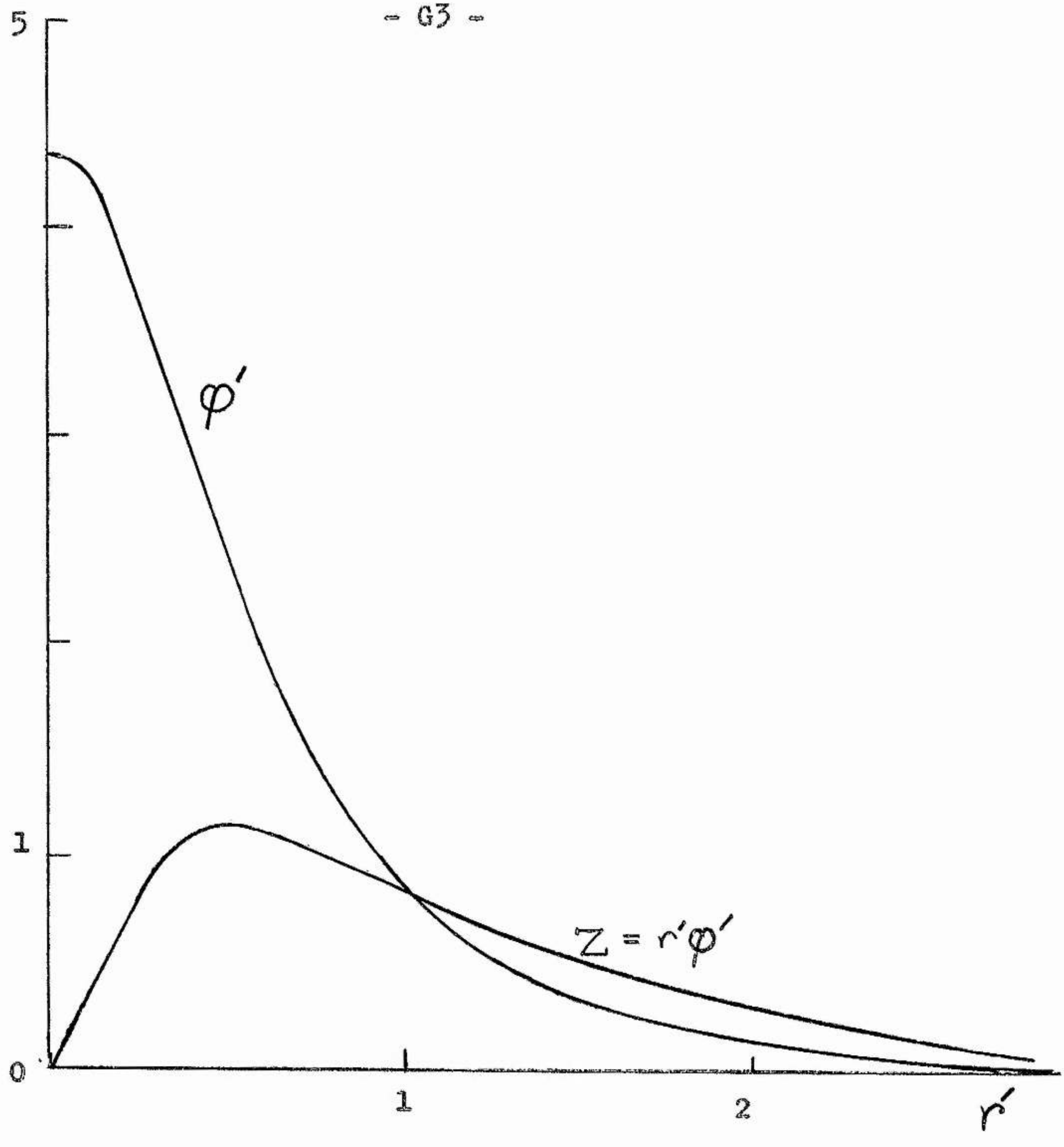


Fig G2

Graph of φ' and $Z = r'\varphi'$ for the lowest-order solution (nodeless) of (1.6) from which it can be seen that Z has a maximum value of 1.2

Appendix H

Significance of having Ω'_r zero

In section 3.3.2 it was remarked that for a given ω', a unique purely imaginary value of Ω' was obtained but that for this solution, a variety of different values of E, P, Q were obtained. It is here shown that the apparent freedom in E, P, Q is a consequence of Ω'_r being zero.

If one puts $\Omega'_r = 0$ in (3.2) then ψ_1 can be expressed in the simpler form

$$\psi_1 = \frac{\hat{\mathfrak{z}}(r)\, e^{\Omega_i t}\, e^{-i\omega t}}{r} \qquad \text{(H1)}$$

where $\hat{\mathfrak{z}}$ replaces $r(\eta + \chi^*)$

If one follows the procedure of section 3.1, and substitutes (H1) into (3.1), and applies the transformation (1.5), then one obtains the coupled equations

$$\left(\frac{d^2}{dr'^2} + A' + 3\varphi_0'^2\right)\hat{\mathfrak{z}}_r - B'\hat{\mathfrak{z}}_i = 0$$
$$\left(\frac{d^2}{dr'^2} + A' + \varphi_0'^2\right)\hat{\mathfrak{z}}_i + B'\hat{\mathfrak{z}}_r = 0 \qquad \text{(H2)}$$

where

$$A' = -1 - \frac{\Omega_i'^2}{(1-\omega'^2)} \qquad B' = \frac{2\omega'\Omega_i'}{(1-\omega'^2)}$$

This has asymptotic form

$$\hat{\mathfrak{z}}_r = \left[\hat{X}\cos(\hat{k}r') + \hat{E}\sin(\hat{k}r')\right]e^{-\Sigma r'}$$
$$\hat{\mathfrak{z}}_i = \left[\hat{X}\sin(\hat{k}r') - \hat{E}\cos(\hat{k}r')\right]e^{-\Sigma r'} \qquad \text{(H3)}$$

where/

where $\hat{k}$ is defined by $\hat{k} = \frac{\beta'}{2\xi}$ and ξ is as given by (3.11) with $\Omega_r' = 0$.

If one solves (H2) and (H3) then for a given ω' one gets the same value of Ω_i' as obtained by solving (3.5) directly, but now one gets a unique value for $\hat{E}$. Further, comparison of the asymptotic form of $(\eta + \chi^*)$ obtained from (3.10) shows that it is, to within a multiplicative constant, the same as that obtained from (H3). It is obvious that if one seeks a priori solutions for which Ω_r' is zero, then this method is easier than solving (3.5) directly.

Appendix I

Proof that Barston's limiting curve and the eigenvalue curve can not be identical.

The equation under examination is

$$\left(\omega_B^2 I - 2i\,\omega_B A - H \right) \xi = 0 \qquad (I1)$$

From this one can express ω_B as

$$\omega_B = \frac{(\xi, iA\xi) \pm \sqrt{(\xi, iA\xi)^2 + (\xi, H\xi)(\xi, \xi)}}{(\xi, \xi)}$$

The eigenvalue curve is for purely complex ω_B, and so we can deduce that

$$(\xi, iA\xi) = 0 \quad \text{and} \quad \omega_{B_i} = \sqrt{\frac{-(\xi, H\xi)}{(\xi, \xi)}} \text{ for this case.}$$

Now the equation of the limiting curve

is $\Omega_i'^2 = \Omega_i'^2(0)\ (1 - \omega^2)$ where $\Omega_i'^2(0)$ = - lowest eigenvalue of the operator H.

i.e. $\left(\omega_{B_i}\right)^2 = \Omega_i'^2(0)$ = - lowest eigenvalue of H.

Now $(\omega_B)^2$ can only equal the lowest eigenvalue of H if ξ equals the corresponding eigenfunction. i.e. ξ is a solution of the equation

$$\left(\omega_B^2 I - H\right) \xi = 0 \qquad (I2)$$

Corresponding to the lowest eigenvalue ω_B^2

But/

But (I1) and (I2) then imply

$$i\,\omega_8\, A\, \xi = 0$$

i.e.

$$\frac{\omega'}{\sqrt{(1-\omega'^2)}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} g+f \\ g-f \end{pmatrix} = 0 \qquad (I3)$$

The lowest eigenfunction of H corresponds to the lowest eigenfunction of $\left(-\frac{d^2}{dr^2} + 1 - 3\varphi_0'^2\right)$ which implies that (g - f) is zero, but that (g + f) is not zero. Hence (I3) will be satisfied only if $\omega' = 0$.
Hence the two curves are not identical.

Appendix J

Proof that there are no complex solutions to (3.6) for any ω' for any $\ell > 1$.

Equation (3.6) can be written in the form (3.12) if H is now defined by

$$\begin{bmatrix} -\frac{d^2}{dr'^2} + \frac{\ell(\ell+1)}{r'^2} + 1 - 3\varphi_0'^2 & 0 \\ 0 & -\frac{d^2}{dr'^2} + \frac{\ell(\ell+1)}{r'^2} + 1 - \varphi_0'^2 \end{bmatrix}$$

Complex eigenvalues Ω' can only occur if ω_B is complex and this in turn can only be so if H has a negative eigenvalue. We show that H can have no negative eigenvalues for any $\ell > 1$. The lowest eigenvalue of H is the lowest eigenvalue of the top diagonal element i.e. the lowest value of Λ to the problem

$$\left[\frac{d^2}{dr'^2} - \frac{\ell(\ell+1)}{r'^2} - 1 + \Lambda - 3\varphi_0'^2\right] Z = 0 \quad \text{(J1)}$$

But it has been shown (Appendix G) that this equation has no square integral eigensolutions for any $\ell > 1$. i.e. there are no solutions to (J1) for $\Lambda < 1$, which implies that H has no negative eigenvalues. There can therefore be no solutions for ω_B complex and hence no solutions for which Ω' is non zero.

Appendix K

Proof that there are no particle-like solutions for $a_o > A_2$ for $0 < B < {}^1/_4$, and none at all for $B > {}^1/_4$

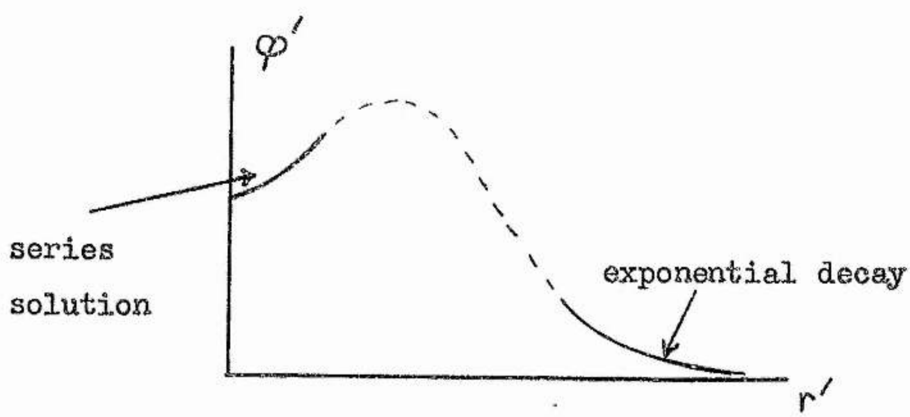
As for the B = 0 case, it is possible to approximate φ' as a power series in r'

$$\varphi' = a_o + a_2 r'^2 + a_4 r'^4 + \ldots\ldots\ldots$$

$$\frac{d\varphi'}{dr'} = 2a_2 r' + \ldots\ldots\ldots\ldots$$

where $a_2 = a_o(1 - a_o^{\,2} + B\, a_o^{\,4})/6$ will be positive if $a_o > A_2$.

Thus for $a_o > A_2$, φ' increases for at least some r' in the neighbourhood of the origin. The asymptotic form of a particle-like solution for large r' is exponential. Thus any particle-like solution must be a combination of these two forms



The nodeless particle-like solution must therefore be of the form shown by dotted curve, which implies φ' must have a maximum for some value/

value of r'. The condition for a maximum is $\frac{d\varphi'}{dr'} = 0 \quad \frac{d^2\varphi'}{dr'^2} = -ve$

When $\frac{d\varphi'}{dr'} = 0$, we have

$$\frac{d^2\varphi'}{dr'^2} = \varphi' - \varphi'^3 + B\varphi'^5 \qquad \text{(K1)}$$

Because φ' must be greater than A_2 at this point $\frac{d^2\varphi'}{dr'^2}$ as given by (K1) must be positive for any B in the range $0 < B < 1/4$. Thus we have an inconsistency implying that such a form of solution cannot exist. The nonexistence of higher node solutions follows as a simple extension of this argument.

We conclude therefore that there are no particle-like solutions for $a_o > A_2$ for any B in the range $0 < B < 1/4$. i.e. $A(k)$ is less than A_2 for all k for all $B > 0$ and $B < 1/4$.

Corollary

There can be no particle-like solutions for $B > 1/4$. For $B > 1/4$, the coefficient a_2 is positive for any positive a_o. Further $\frac{d^2\varphi'}{dr'^2}$ as given by (K1) is always +ve, and thus it is not possible to have particle-like solutions for any a_o in this case.

Appendix L

Bounds to the eigenvalue $\Omega_i'(0)$

The lowest eigenvalue of the operator H can be bounded as follows. Let us consider the top diagonal element, denoting the lowest eigenvalue of this operator by $-\Lambda$. Then the problem reduces to solving the equation

$$\left\{ \frac{d^2}{dr'^2} - \gamma^2 + 3\varphi_0'^2 - 5B\varphi_0'^4 \right\} z = 0 \qquad \gamma^2 = 1 + \Lambda \tag{L1}$$

This equation is equivalent to finding the eigenvalue $\Omega_i'(0)$ of (7.7) if one identifies γ^2 with $1 + \Omega_i'^2(0)$ where $\Omega_i'(0)$ is the maximum value of Ω_i' when $\omega' = 0$. Using a now familiar argument, any solution (square integrable) of (L1) must have a point of inflexion at which the eigenfunction is non zero. This implies

$$-\gamma^2 + 3\varphi_0'^2 - 5B\varphi_0'^4 = 0 \tag{L2}$$

at such a point. Now (L2) can be considered as an equation in $\varphi_0'^2$ with solution

$$\varphi_0'^2 = \frac{3 \pm \sqrt{9 - 20B\gamma^2}}{10B} \tag{L3}$$

It is a requirement that $\varphi_0'^2$ be real and thus we can immediately say that

$$\gamma^2 < \frac{9}{20B} \qquad \text{for } B > 0$$

i.e. $\Omega_i'^2(0) < (\frac{9}{20B} - 1)$

Appendix M

Definition of $\mathcal{G}_1(b)$, $\mathcal{G}_2(b)$

When disturbances Ψ_1 were applied to Ψ_{oS}, Ψ_{oR}, Ψ_{ou} this was done numerically. In consequence one had to specify the disturbances numerically at a set of discrete points. It was convenient to have a set of such disturbances which one could apply at various distances from the particle, and from which one could construct a whole variety of other disturbances. In chapter 8 we used only the disturbances $\mathcal{G}_1$ and $\mathcal{G}_2$. Rather than give a list of the numeric values, we have shown the disturbances graphically in figs M1 and M2 respectively. The following notation is operative.

By $\mathcal{G}_1(b)$ we mean the function

$$\mathcal{G}_1 = 0 \qquad \text{for} \quad \rho < b$$
$$\mathcal{G}_1 = f_1(\rho - b) \qquad \text{for} \quad \rho > b$$

where $f_1(\rho)$ is plotted in fig M1.

A similar definition holds for $\mathcal{G}_2(b)$

$$\mathcal{G}_2 = 0 \qquad \text{for} \quad \rho < b$$
$$\mathcal{G}_2 = f_2(\rho - b) \qquad \text{for} \quad \rho > b$$

where $f_2(\rho)$ is plotted in fig M2.

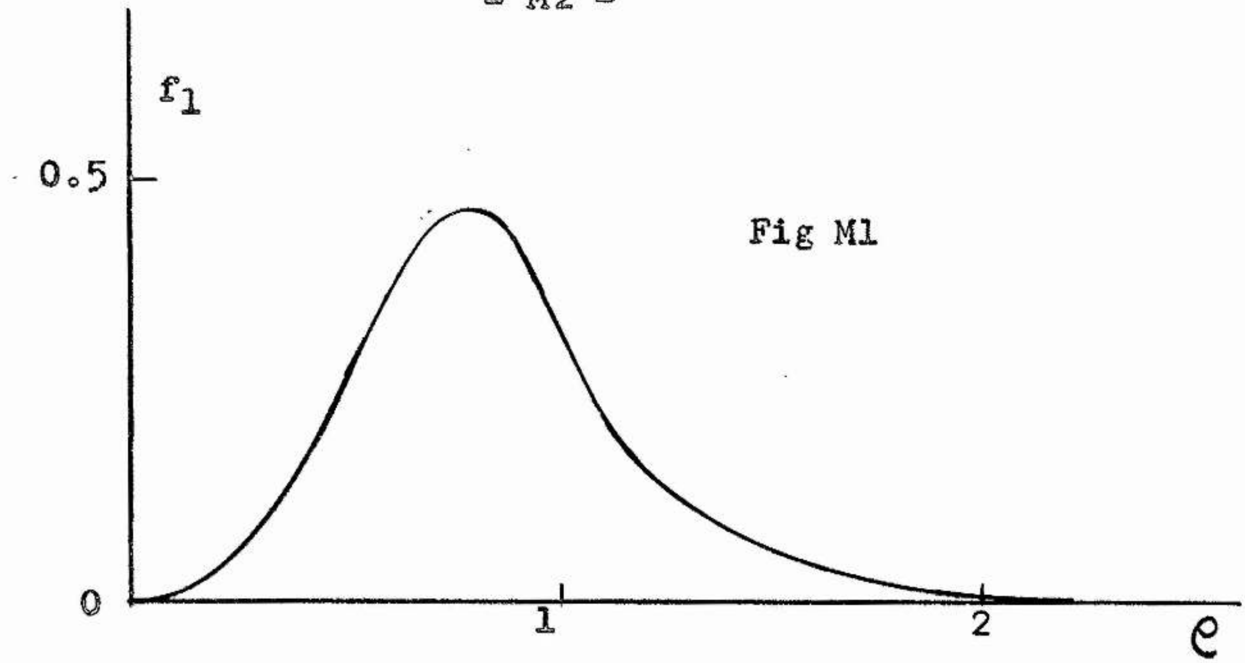


Fig M1

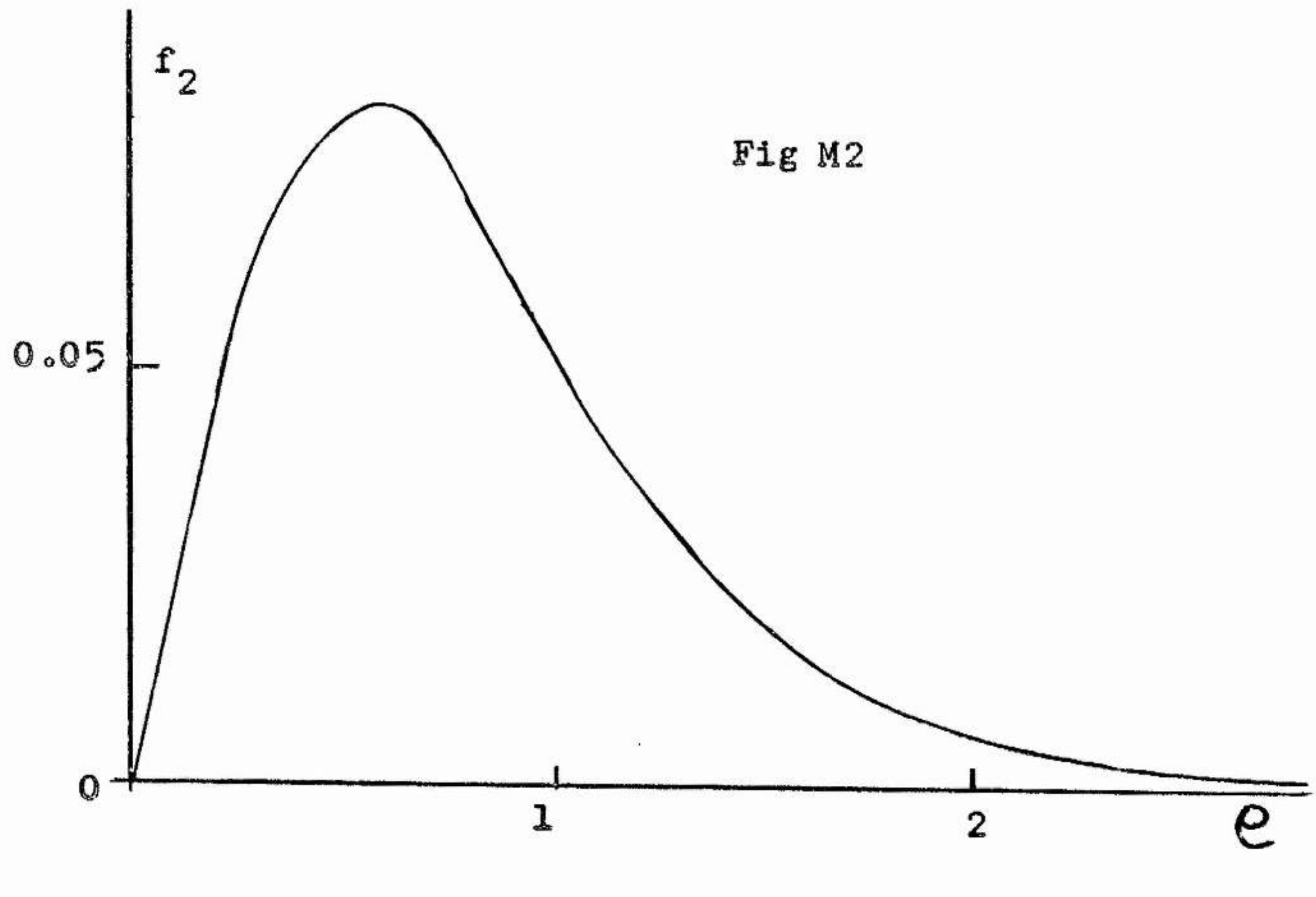


Fig M2

Plots of $f_1(\rho)$ and $f_2(\rho)$ against the reduced radial distance ρ.

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